Developing Multiplicative Thinking -

Developing Multiplication Strategies
with Bonny Davenport
Welcome!

Your host

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Good News!
The KCM is hard at work to ensure Kentucky teachers have access to innovative professional development from home.

Through the newly launched KCM Virtual site, mathematics teachers from all grade levels will have access to live zoom meetings, video records and corresponding materials. Read more.

Developing Multiplicative Thinking - Apr. 27 - May 1

Focus on Fractions - May 4 - May 8

And the math continues with these sessions under development:

Focus on Geometry - May 11 - May 15

More Multiplicative Thinking - May 18 - May 22

Focus on Measurement & Data - May 26 - May 29
Today’s Agenda

• Standards
• Let’s Do Math!
• Research
• Derived Facts For Multiplication
  • Doubling
  • Break Apart
  • Adding a Group
  • Subtracting a Group
  • Near Squares
• Properties of Multiplication
• Points to Consider
## Standards

### Operations and Algebraic Thinking

#### Standards for Mathematical Practice

<table>
<thead>
<tr>
<th>Standards</th>
<th>Clarifications</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MP.1</strong>. Make sense of problems and persevere in solving them.</td>
<td><strong>MP.5</strong>. Use appropriate tools strategically.</td>
</tr>
<tr>
<td><strong>MP.2</strong>. Reason abstractly and quantitatively.</td>
<td><strong>MP.6</strong>. Attend to precision.</td>
</tr>
<tr>
<td><strong>MP.3</strong>. Construct viable arguments and critique the reasoning of others.</td>
<td><strong>MP.7</strong>. Look for and make use of structure.</td>
</tr>
<tr>
<td><strong>MP.4</strong>. Model with mathematics.</td>
<td><strong>MP.8</strong>. Look for and express regularity in repeated reasoning.</td>
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### Cluster: Multiply and divide within 100.

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<tr>
<td>KY.3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division or properties of operations. <strong>MP.2, MP.8</strong></td>
<td>Students determine multiplication and division strategies efficiently, accurately, flexibly and appropriately. Being fluent means students choose flexibly among methods and strategies to solve contextual and mathematical problems, they understand and explain their approaches and they produce accurate answers efficiently. Knowing 8 x 5 = 40, one knows 40 ÷ 5 = 8.</td>
</tr>
<tr>
<td>Note: Reaching fluency is an ongoing process that will take much of the year.</td>
<td></td>
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</tbody>
</table>

#### Attending to the Standards for Mathematical Practice

By studying patterns and relationships in multiplication facts, students develop fluency for multiplication facts (**MP.8**). For example, students notice 4 x 6 is equivalent to 2 x 2 x 6 (doubling strategy). They know 9 facts can be found by thinking of the other factor x 10 and subtracting one group. For example, recognizing 9 x 8 is equivalent to 10 x 8 – 8. For each fact, the student thinks, “What reasoning strategy can I use that is more efficient than skip counting?” (**MP.2**).
### Standards

#### Operations and Algebraic Thinking

**Standards for Mathematical Practice**

- **MP.1.** Make sense of problems and persevere in solving them.
- **MP.2.** Reason abstractly and quantitatively.
- **MP.3.** Construct viable arguments and critique the reasoning of others.
- **MP.4.** Model with mathematics.
- **MP.5.** Use appropriate tools strategically.
- **MP.6.** Attend to precision.
- **MP.7.** Look for and make use of structure.
- **MP.8.** Look for and express regularity in repeated reasoning.

**Cluster: Understand properties of multiplication and the relationship between multiplication and division.**

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<td><strong>KY.3.OA.5</strong> Apply properties of operations as strategies to multiply and divide. <strong>MP.3, MP.4</strong></td>
<td>Students need not use formal terms for these properties. If $6 \times 4$ is known, then $4 \times 6 = 24$ is also known (Commutative property of multiplication). $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$ (Associative property of multiplication). Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find $8 \times 7$ as $8 \times (5+2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$ (Distributive property).</td>
</tr>
<tr>
<td><strong>KY.3.OA.6</strong> Understand division as an unknown-factor problem. <strong>MP.2</strong></td>
<td>Find $32 \div 8$ by finding the number that makes $32$ when multiplied by $8$.</td>
</tr>
</tbody>
</table>

**Attending to the Standards for Mathematical Practice**

Students use strategies beyond skip counting to solve multiplication problems. They decide how to use known facts to solve facts like $6 \times 9$. Students use strategies like Adding a Group, thinking $9$ groups of $6$ ($45$) plus one more group ($54$) and Subtracting a Group, thinking $9 \times 6$ and reasoning $10$ groups of $6$ ($60$) minus one group of $6$ ($54$) (**MP.7**). Students explain their selected reasoning strategy and listen and critique other students’ strategies, considering which strategies make sense and are efficient (**MP.3**). Students think about $84 \div 4$ as, “How many sets of $4$ can be made from $84$ items?” or “How many in a group, if there are $84$ items and $4$ groups?” and use this relationship to solve the problem (**MP.2**).
Let’s Do Some Math!

How can we figure out the cost of buying 4 hats?

$16

https://www.origoeducation.com/blog/doubling-strategy-for-multiplication/
Five Fundamentals of Fact Fluency

#1: Mastery must focus on fluency!
#2: Fluency develops in three phases.
#3: Knowing foundational facts must precede derived facts.
#4: Timed tests do not assess fluency
#5: Students need substantial and enjoyable practice.

Mastery Must Focus on Fluency

Figure 1.1. What Procedural Fluency Is and What It Looks Like

The four components (bolded) are interrelated. Appropriate strategy selection is required for efficiency and flexibility.

Fluency Develops in Three Phases

Phase 1: Counting
Student counts with objects or mentally.
Example: Solving $6 \times 4$ by drawing 6 groups of 4 dots and counting the dots.

Phase 2: Deriving
Uses reasoning strategies based on known facts.
Example: Solving $6 \times 4$ by thinking $5 \times 4 = 20$ and adding one more group of 4.

Phase 3: Mastery
Efficiently produces answers.
Example: Knows $6 \times 4 = 24$

Foundational Facts Must Precede Derived Fact Strategies

Doubling (4s, 6s, 8s)

Look for an even factor. Find the fact for half of that factor, then double it.

Example: I don't know 6 x 8 so I think “3x8=24” and double that to get 48.

How might we solve 7x6?
Multiplication Stories

Carefully sequenced stories can encourage the use of halving and doubling. The area model can help students visualize how doubling one of the factors leads to doubling the area, or product.

Sequenced Quick Looks for Doubling

Games for **Doubling** and Learning Facts

**Game: Switch**

(2 players)

**Materials:**
- One piece of paper, shared by all players
- 5 or 6 unique game pieces per player (e.g., colored counters or heads/tails of coins)
- Deck of cards, with Kings and Jacks removed; Queens = 0; Aces = 1

**How to play:**
1. Write the multiples of 4 (0, 4, 8, 12, 16, 20, 24, 28, 32, 36, and 40) on a piece of paper (mix up the order of products to discourage skip counting as you play - see below).

2. Shuffle the deck of cards and place them face down in a draw pile.

3. Players take turns drawing one card and multiplying the drawn card by 4. The player places his/her game piece above the resulting product. For example, if a player draws the number 7, the player multiplies 7 by 4 and places a game piece above 28. Players share their strategy aloud (e.g., “I doubled 7 to get 14, and doubled again to get 28.”).

4. If a player’s resulting product already has an opponent’s game piece on it, that player gets to SWITCH, placing their own game piece above that number, and returning their opponent’s game piece. If the player’s resulting product is one that they themselves already have, they lose that turn.

5. **To win:**
   - Option 1: First to get 5 game pieces on the board
   - Option 2: Have the most game pieces when the board is completely covered (as illustrated here).

*Pick an option and play best 3 out of 5!*

**More ways to play:**
Play with any fact set by creating a game board with the multiples of that set (e.g., \( \times 7 \)).
Play with three people (use three different colors of counters or three different coins).
**Game: Fixed Factor War** (Game 32, p. 88, Math Fact Fluency)  
(2 players)

**Materials:**
- Deck of cards, with Kings and Jacks removed. Queens = 0; Aces = 1.

**How to Play:**
1. Find a 4 in the deck and place it between the two players (or 6 or 8) face up. That number is the fixed factor.
2. Deal the rest of the cards equally, face down.
3. Each player takes a turn to flip over the top card of his/her pile of cards. The player must state the product of the “fixed” factor card and the card they flipped, and share how they know (see example below).
4. The player who correctly states the greater product in the round gets both players’ cards. (The “middle” fixed factor card stays.)
5. If there is a tie, a “war” is declared, and players repeat the process, with the winner taking all played cards.
6. The player with the most cards wins when time is up.

**More ways to play:** Use different Fixed Factors (e.g., a 6). Play Factor War – No fixed factor, each player draws 2 cards. Play with addition, too! (Fixed Addend War or Addend War)

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**Fixed Factor Card**  
(Does not change.)

**Player**

Nicolas turns over a 3. He says, “Twelve. I doubled 3 to get 6 and doubled again to get 12.”

**Player**

MacKenna turns over a 5. She says, “Twenty. I just know my 5s, 5 times 4 equals 20.”
Break Apart (3s, 4s, 6s, 7s, 8s, 9s)

Partition one of the factors into a convenient sum of known facts, find the two known facts, and combine the products.

Example: I don’t know 7x6. I break the 7 into 2 and 5, because I know 2x6 and 5x6. Then I add 12 and 30 to get 42.

How might we solve 6x8?
When students begin to break apart numbers, using a representation is key to keeping track of their process.

Streets, Avenues and Stoplights

Horizontal toothpicks = streets
Vertical toothpicks = avenues
Intersections = stoplights

How might a student figure out the number of stoplights needed for this town?
How Close to 100?

This has become one of our most popular tasks and we are hearing about all sorts of creative adaptations. Some youcubians have made grids of 400 and added dice, others have adapted it to let the grid represent 100%. Please post how you use this task with your students.

Task Instructions

- This game is played in partners. Two children share a blank 100 grid.
- The first partner rolls two number dice.
- The numbers that come up are the numbers the child uses to make an array on the 100 grid.
- They can put the array anywhere on the grid, but the goal is to fill up the grid to get it as full as possible.
- After the player draws the array on the grid, she writes in the number sentence that describes the grid.
- The second player then rolls the dice, draws the number grid and records their number sentence.
- The game ends when both players have rolled the dice and cannot put any more arrays on the grid.
How Close to 100?
How Close to 100?

1. \( \frac{3}{2} \times 2 = 6 \)
2. \( \frac{2}{2} \times 3 = \frac{6}{2} \)
3. \( \frac{2}{2} \times 6 = 12 \)
4. \( \frac{2}{2} \times 6 = 30 \)
5. \( \frac{6}{2} \times 2 = 12 \)
6. \( \frac{5}{6} \times 6 = 30 \)
7. \( \frac{3}{2} \times 6 = 18 \)
8. \( \frac{4}{2} \times 6 = 36 \)
9. \( \frac{2}{1} \times 1 = 2 \)
10. \( \frac{6}{2} \times 2 = 12 \)

How Close to 100?

1. \( \frac{3}{2} \times 2 = 6 \)
2. \( \frac{6}{2} \times 5 = \frac{30}{2} \)
3. \( \frac{2}{2} \times 4 = 8 \)
4. \( \frac{3}{2} \times 2 = 6 \)
5. \( \frac{2}{1} \times 1 = 2 \)
6. \( \frac{2}{1} \times 4 = 8 \)
7. \( \frac{3}{2} \times 9 = 15 \)
8. \( \frac{3}{2} \times 1 = \frac{3}{2} \)
9. \( \frac{3}{2} \times \_ = \_ \)
10. \( \_ \times \_ = \_ \)
Adding a Group (3s, 6s)

Start with a nearby 2s, 5s or 10s fact, then add the group.

Example: I don’t know 6x7, but I do know my 5s, so I can first find 5x7. I know 5 groups of 7 is 35. I have to add one more group of 7 to 35 and that equals 42.

How might we solve 3x8?
Stories Provide Context

Sequenced number stories help students make sense of the add a group strategy. A sequenced number story comes in two parts, with the first part involving known facts and the second part providing a change in the story so that another group is added.

Quick Sketches for Adding a Group Strategy

6 x 7 means 6 groups of 7
5 groups of 7 equal 35
35 + 7 = 42
6 groups of 7 = 42

The equal groups meaning of multiplication must remain at the forefront of strategy work. Without that solid foundation, students may be able to start with the helper fact but become confused on what to do next.

Multiplication
This week your student will work with the build-up strategy for multiplication. This strategy can be used anytime you multiply a number by 6 by building up from a known 5’s fact. For example, 5 groups of 8 is 40, so 6 groups of 8 must be 8 more, so the answer is 48.
Watch the ORIGO ONE video about the build-up strategy before working with your student (also available in Spanish). This will help you to assist your student as they work through the activities this week. Encourage your student to look for patterns as they multiply by 6.

<table>
<thead>
<tr>
<th>Monday — Watch and Talk</th>
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<tbody>
<tr>
<td>Tuesday — Hands-on Math</td>
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<tr>
<td>Wednesday — Problem-solv</td>
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<tr>
<td>Thursday — Game Day</td>
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<tr>
<td>Friday — Practice</td>
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<table>
<thead>
<tr>
<th>6 × 1</th>
<th>6 × 2</th>
<th>6 × 3</th>
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<tbody>
<tr>
<td>6 × 4</td>
<td>6 × 5</td>
<td>6 × 6</td>
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<tr>
<td>6 × 7</td>
<td>6 × 8</td>
<td>6 × 9</td>
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</table>

<table>
<thead>
<tr>
<th>36</th>
<th>18</th>
<th>42</th>
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<tbody>
<tr>
<td>12</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>30</td>
<td>48</td>
<td>54</td>
</tr>
</tbody>
</table>
Subtracting a Group (9s, 4s)

Start with a nearby 2s, 5s or 10s fact, then subtract the group.

Example: I don’t know 8x7, but I do know my 10s facts, so I can first find 10x7. I know ten groups of 7 is 70. That is two groups too many. I have to subtract two groups of 7 from 70 and that is 70-14=56. So, 8x7=56

How might we solve 9x4?
Sequenced Number Story for Subtracting a Group

1. Amanda is stacking cans on the shelf at the grocery store. She has room for 10 rows of cans. She can put 6 cans in each row. How many cans can Amanda stack on the shelf?

2. A customer bought a row of cans. Now Amanda only has 9 rows with 6 cans in each row. Use what you already know to figure out how many cans Amanda has on the shelf now.
Multiplication
This week will focus on the build-down strategy for multiplication.
This strategy can be used when multiplying a number by 9. If you have time, watch the ORIGO ONE video about the build-down strategy before working with your student (also available in Spanish).
This will help you to assist your student as they work through the activities for this week. Encourage your student to look for patterns as they multiply by 9.

Monday — Watch and Talk
Tuesday — Hands-on Math
Wednesday — Problem-solving
Thursday — Game day
Friday — Practice day

Add the cards then multiply by 9.
Near Squares

Look for a nearby square. Find that fact and add on or subtract off the extra group.

Example: I don’t know 7x6. I use 6x6 and add one more 6 to get 42.

I don’t know 7x6. I use 7x7 and subtract one more 7 to get 42.

How might we solve 8x7?
Squares/ Near Squares Kaboom

Materials: Popsicle sticks with an individual fact from the list below on each stick. Write Kaboom on three or more additional popsicle sticks. Optional: calculator/multiplication chart to check answers.

Directions: Put all the popsicle sticks face down so students cannot see what is written on them. Here is how to play:

1. First student pulls out a popsicle stick.

2. The student identifies the answer and explains his/her thinking to the group/partner. If the answer is correct, the student gets to keep the popsicle stick. If the student answers it incorrectly, the stick must go back in the cup.

3. The students continue around the circle, or with a partner, selecting one popsicle stick at a time and answering their question.

4. Any student who pulls a KABOOM! stick must place all the popsicle sticks they have accumulated back into the cup, leaving them with zero.

5. The student with the most sticks after 5 rounds, time is up, etc. is the winner.

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<tbody>
<tr>
<td>1 x 1</td>
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<td>9 x 8</td>
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</table>
**Multiplication**

Using the multiplication problem at the top of each chart, create a representation of the problem that matches the labels in each of the four boxes.

### Card Set A

| 3 x 4 | 4 x 8 |

### Card Set B

| 9 | 3 |
Tiling With Numbers

I can combine two rectangles to make a larger rectangle and find the total tiles needed to cover each part and the larger rectangle.

Materials: Rectangle Cards, 1 recording sheet per player

Directions:
1. Place all rectangle cards face down in a pile.
2. With a partner take turns selecting a rectangle card without looking. Each player should place their rectangle cards face up in front of them.
3. When you have two cards that can go together to make a larger rectangle, place them together and find the total number of tiles needed to cover both parts.
4. Write the facts for each match on your recording sheet. The “larger rectangle” is not a separate piece – just a composite of the two smaller rectangles.
5. When all the cards are gone, the player with the most 2-piece rectangles wins.
Materials:
• Rectangles with dimensions labeled
• Recording sheets (1 per player)
Game Play

On a player’s turn, the player will draw 1 rectangle and place it on the table.

If possible, the player will create a larger rectangle by pairing two cards. Otherwise, the player will pass.
Game Play

Record the multiplication expressions and area on the recording sheet.

Winner is player who creates the most large rectangles.
Commutative Property of Multiplication

8 x 2 = 2 x 8

Changing the order of the factors does not change the product.
Associative Property of Multiplication

(6 x 2) x 2

6 x (2 x 2)

Changing the groupings of the factors does not change the product.
A factor can be decomposed into addends and the addends can each be multiplied by the other factor to find partial products, and then those partial products can be added to find the total product.
## Properties of Multiplication

Students need meaningful practice to move from fluency to mastery. CCSSM 3.OA.B.5: “Apply properties of operations as strategies to multiply and divide.”

<table>
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<tr>
<th>Property</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>Commutative property of multiplication</td>
<td>Important to all facts. If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. This cuts the learning of facts in half.</td>
</tr>
<tr>
<td>Associative property of multiplication</td>
<td>Used in derived facts, like doubling. A student sees $6 \times 9$ and thinks $(2 \times 3) \times 9$, which is the same as $2 \times (3 \times 9)$, which is $2 \times 27$, $54$.</td>
</tr>
<tr>
<td>Distributive property of multiplication over addition</td>
<td>A student realizes that $8 \times 7 = 8 \times (5 + 2)$ and uses this to find the answer, thinking $(8 \times 5) + (8 \times 2) = 40 + 16 = 56$.</td>
</tr>
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</table>

Strategy Focused Game Play: Aligning Strategies With Games

- What set of facts will be the focus?
- What strategy aligns with that set of facts?
- What games will encourage the use of this strategy and focuses on this fact set?
- How can I incorporate sequenced story problems?
- What Quick Looks/ Number Talks might I use to support student learning?
Time to Share!

Anything SQUARE with your way of thinking?

A POINT (or 3!) you would like to make?

Anything still CIRCLING in your mind?
Upcoming Sessions

APRIL 27 - MAY 1  
2:00-2:30 PM EST

Developing Multiplicative Thinking!

Monday, April 27 - Foundations of Multiplicative Thinking

Tuesday, April 28 - Sequence of Multiples

Wednesday, April 29 - Structuring Numbers Multiplicatively

Thursday, April 30 - Developing Multiplication Strategies

Friday, May 1 - Monitoring and Assessing Multiplication
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