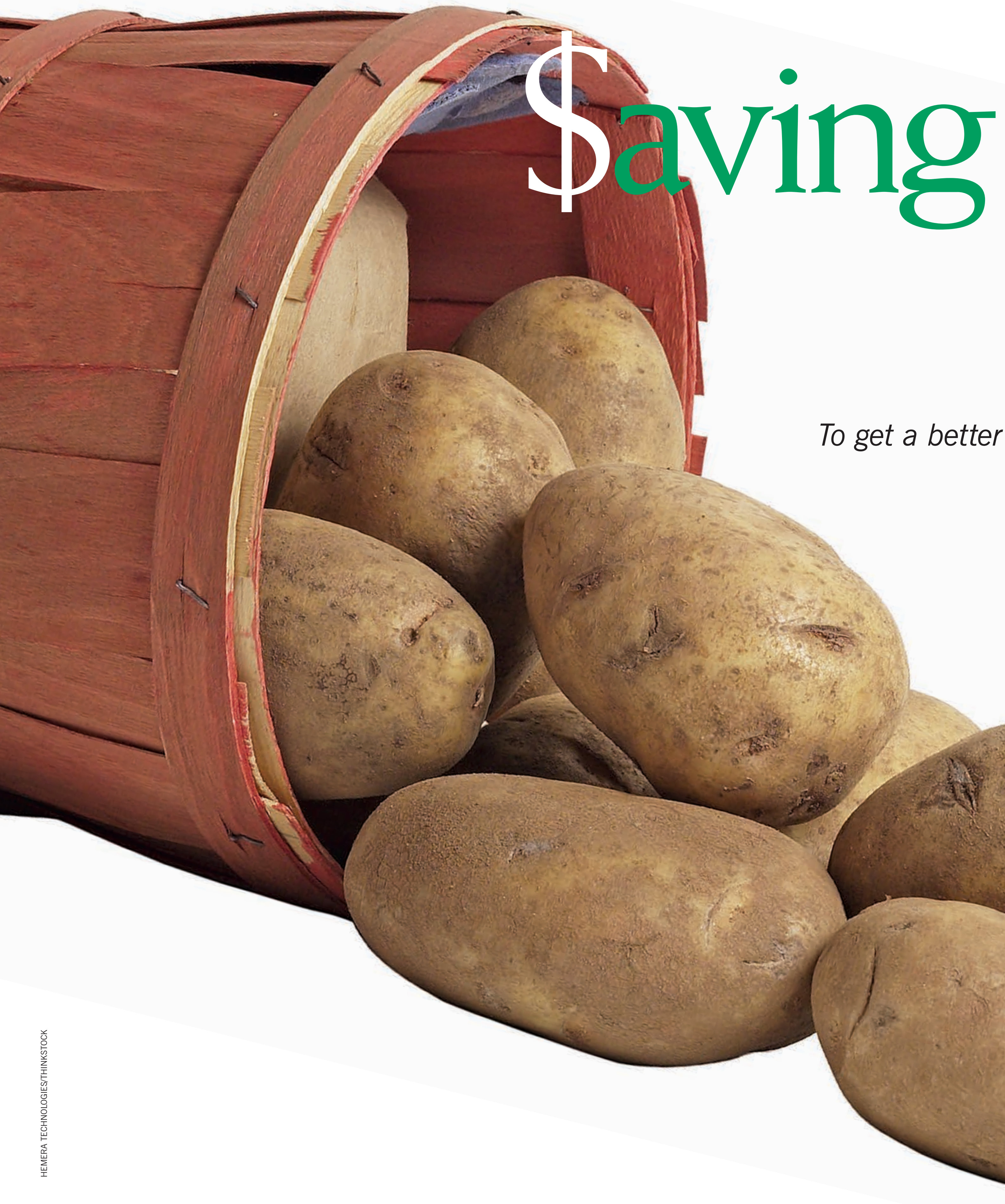
# $aving



*To get a better*

HEMERA TECHNOLOGIES/THINKSTOCK

# Money

# USING PROPORTIONAL REASONING

*buy, students employ various strategies to compare grocery prices.*

Jessica A. de la Cruz and Sandra Garney

How would your middle school students solve this missing value problem:

> If 4 pounds of potatoes cost $6.00, how much would 10 pounds of potatoes cost?

Would they be more likely to apply the cross-multiplication algorithm, as opposed to simpler multiplicative reasoning approaches? Although cross multiplication results in a correct answer, students using this method do not necessarily understand proportionality. Rather than the more commonly used missing-value problems, like the previous example, we suggest posing comparison problems to help students recognize the underlying multiplicative relationship that exists within a proportion.

Consider this comparison problem:

> Which is a better price for potatoes: $1.29 for 10 pounds or $4.99 for 20 pounds?

Here, a factor-of-change strategy is intuitive (if the number of pounds doubles, then so should the price), and it emphasizes the multiplicative relationship between the two ratios. Moreover, cross multiplication is likely to be unsuccessful in determining the better price because students must interpret the relationship between the cross products. **Figure 1** presents an example of cross multiplication when applied to compare ratios.

Many teachers would agree that once cross multiplication is introduced, their students tend to apply it by rote, abandoning all previously learned proportional reasoning strategies. Although cross multiplication is typically the most emphasized strategy in textbooks for solving missing-value proportion problems, many researchers believe that an overemphasis of this strategy is the root of students' difficulties with proportional reasoning. One study even found that students who were taught the cross multiplication strategy were actually

**Fig. 1** Using a cross-multiplication strategy to compare two prices yields two products that are difficult to interpret in terms of the original scenario.

$$\frac{\$2.50}{2\text{ lb.}} \stackrel{?}{=} \frac{\$3.25}{3\text{ lb.}}$$

$$(\$2.50)\times(3\text{ lb.}) \stackrel{?}{=} (\$3.25)\times(2\text{ lb.})$$

less successful when solving proportion problems than students who were never taught the algorithm (Fleener, Westbrook, and Rogers 1993). Additionally, repetitive application of cross multiplication without knowledge of other proportional reasoning strategies is not enough to be considered proportional reasoning (Cramer, Post, and Currier, 1993; Fleener, Westbrook, and Rogers 1993). Students should fully develop more intuitive strategies, such as factor of change or unit rate strategies, before being introduced to cross multiplication. These intuitive strategies help students better understand the multiplicative relationship between proportional ratios. However, many textbooks heavily emphasize cross multiplication and leave a gap where teachers must develop other ways to foster the creation and use of different proportional reasoning strategies.

**Fig. 2** The comparison task and anticipated strategies for an integer factor of change (× 2) problem are illustrated.

1. Which is the better deal for potatoes?


A


B

**Factor of Change (× 2)**

$$\text{A: } \frac{\$1.29}{10\text{ lb.}} \underset{\times 2}{\overset{\times 2}{=}} \frac{\$2.58}{20\text{ lb.}} \qquad \text{B: } \frac{\$4.99}{20\text{ lb.}}$$

**Unit Rate**

$$\text{A: } \frac{\$1.29}{10\text{ lb.}} \underset{\div 10}{\overset{\div 10}{=}} \frac{\$0.13}{1\text{ lb.}} \qquad \text{B: } \frac{\$4.99}{20\text{ lb.}} \underset{\div 20}{\overset{\div 20}{=}} \frac{\$0.25}{1\text{ lb.}}$$

HEMERA TECHNOLOGIES/THINKSTOCK

It is beneficial for students to discover intuitive strategies, as opposed to the teacher presenting strategies to them. Certain proportional reasoning tasks are more likely to elicit intuitive strategies than other tasks. The strategies that students are apt to use when approaching a task, as well as the likelihood of a student's success or failure solving it, are influenced by that task's context and numerical structure (de la Cruz 2013). Thus, teachers can encourage the development of particular strategies by carefully selecting the tasks that students will complete. Furthermore, implementing the Five Practices (Smith et al. 2009) can assist teachers in structuring the whole-class sharing of student-generated strategies in an organized and purposeful way. Considering the effects that task characteristics can have on strategy choices, we designed the Better Buy Lesson, which we describe here.

## THE FIVE PRACTICES MODEL

Smith and colleagues (2009) present a model to support and prepare teachers to incorporate students' thinking into classroom discussion. Focusing on the Five Practices helps teachers by limiting the in-the-moment decisions that are sometimes frightening aspects of

**Fig. 3** The comparison task and anticipated strategies for an integer factor of change (× 10) problem, with the unit rate provided, are explored next.

2. Which is the better deal for potatoes?


A


B

**Unit Rate**

$$\text{A: } \frac{\$1.29}{10 \text{ lb.}} \overset{\div 10}{=} \frac{\$0.13}{1 \text{ lb.}} \qquad \text{B: } \frac{\$0.69}{1 \text{ lb.}}$$

**Factor of Change (× 10)**

$$\text{A: } \frac{\$1.29}{10 \text{ lb.}} \qquad \text{B: } \frac{\$0.69}{1 \text{ lb.}} \overset{\times 10}{=} \frac{\$6.90}{10 \text{ lb.}}$$

student-centered teaching. Moreover, it better prepares teachers to highlight the facets of students' thinking that tie specifically to the instructional goal. The Five Practices include the following: (1) anticipating, (2) monitoring, (3) selecting, (4) sequencing, and (5) connecting. When planning the Better Buy Lesson, we chose challenging mathematical tasks while anticipating the strategies that students would use when solving. Next, we selected the strategies we aimed to share in the discussion portion of the lesson by considering our ultimate instructional goals. Then we predicted how we would sequence the shared strategies, with the understanding that this sequence may be adapted, depending on what we observed when monitoring the classwork. Finally, we planned how we would connect the shared strategies to each other and to our instructional goals.

## THE BETTER BUY ACTIVITY

Students, working in pairs, were asked to determine the better deal when given two different prices and quantities for similar items found in competing grocery store ads. They were instructed to use any strategy that they could fully explain to the class. In total, there were four comparison tasks (see **figs. 2–5**). Before the lesson, we chose each task carefully, anticipating strategies we thought students would use and after analyzing each task's numerical structure.

### *Choosing the Comparison Tasks and Anticipating Strategies*

First, we predicted that students would have little success applying cross multiplication to compare the ratios, which is consistent with Singh's (2000) research. When the rates being compared are not proportionally related, interpreting the cross products is difficult (see **fig. 1**). It is clear from the unequal cross products that the ratios are not equivalent; however, it is not clear which one is the better buy. This meant that students would likely employ alternative strategies.

Second, we had two goals in mind when analyzing and selecting the four comparisons: To encourage flexible use of several proportional reasoning strategies and to emphasize the multiplicative nature of proportional ratios. Depending on the strategy, we chose particular numerical structures known to influence different problem-solving approaches (Tjoe and de la Torre 2013)

According to Lesh, Behr, and Post (1987), the presence of an integer factor of change between the ratios increases the likelihood that students would apply a factor of change strategy, also referred to in the literature as a building up through multiplication strategy (Steinthorsdottir and Sriraman 2009). The following comparison would likely encourage the use of a factor of change strategy: \$15.00 for 4 pounds of dog food at store A versus \$78.00 for 24 pounds at store B. At store A, we can determine the price for 24 pounds using a factor of change of 6:

$$\frac{\$15.00}{4 \text{ lb.}} = \frac{\$90.00}{24 \text{ lb.}}$$

The presence of an integer factor of change within one of the ratios (i.e., an integer unit rate), coupled with the absence of an integer factor of change between the ratios, encourages students to apply unit rate strategies. For instance, \$15.00 for 5 pounds of dog food at store A versus \$76.00 for 19 pounds would likely be solved using a unit rate strategy: At store A,

$$\frac{\$15.00}{5\text{ lb.}} = \frac{\$3.00}{1\text{ lb.}};$$

at store B,

$$\frac{\$76.00}{19\text{ lb.}} = \frac{\$4.00}{1\text{ lb.}}.$$

**Figures 2–5** present the four comparison tasks we created, highlight the numerical structure for each, and list the approaches that we anticipated students would use. When designing the activity, we aimed to have students perform these strategies: factor of change, unit rate, common denominator, and combination strategies. We looked for these specific strategies when we monitored the activity.

***Monitoring Students' Work***

According to Smith et al. (2009), teachers should monitor their students' thinking and strategies as they work to productively determine who should share and what should be shared in class discussion. Without careful monitoring and selecting, the discussion can turn into a "show and tell" of disconnected strategies and may not deepen students' understandings. **Figure 6** depicts the table we used to record our assessments throughout the monitoring process. It also indicates decisions that resulted when we anticipated students' approaches while also considering our instructional goal. We included an additional row at the bottom of the table to capture any unforeseen strategies as well as note incorrect additive approaches.

**Fig. 4** This comparison task and anticipated strategies show a problem with no integer factor of change.

3. Which is the better deal for 12 packs of Coca-Cola?


A


B

**Unit Rate**

$$A: \overset{\div 4}{\underset{\div 4}{\frac{\$9.00}{4\text{ packs}} = \frac{\$2.25}{1\text{ pack}}}} \qquad B: \overset{\div 10}{\underset{\div 10}{\frac{\$22.50}{10\text{ packs}} = \frac{\$2.25}{1\text{ pack}}}}$$

**Common Denominator**

$$A: \overset{\times 5}{\underset{\times 5}{\frac{\$9.00}{4\text{ packs}} = \frac{\$45.00}{20\text{ packs}}}} \qquad B: \overset{\times 2}{\underset{\times 2}{\frac{\$22.50}{10\text{ packs}} = \frac{\$45.00}{20\text{ packs}}}}$$

**Combination of Buildup and Reduction**

$$A: \overset{\times 2}{\underset{\times 2}{\frac{\$9.00}{4\text{ packs}} = \frac{\$18.00}{8\text{ packs}}}} \qquad B: \frac{\$22.50}{10\text{ packs}}$$

$$\overset{\div 2}{\underset{\div 2}{\frac{\$9.00}{4\text{ packs}} = \frac{\$4.50}{2\text{ packs}}}}$$

$$\frac{\$9.00}{4\text{ packs}} = \frac{\$18 + \$4.50}{8 + 2\text{ packs}} = \frac{\$22.50}{10\text{ packs}}$$

***Selecting, Sequencing, and Connecting Students' Work***

After monitoring the students' work on the four comparison tasks and referencing our monitoring tool, specific groups were selected to share their strategies with the class. A pair who used long division to calculate the unit prices per pound of potatoes, in problem number 1, was asked to share first. Their work is depicted in **figure 7a**. Next, a pair was chosen to share their

**Fig. 5** The task and anticipated strategies for a problem involved comparing three ratios.

4. Which is the better deal for paper towels?


**Unit Rate**

÷ 8

$$A: \frac{\$8.99}{8\text{ rolls}} = \frac{\$1.12}{1\text{ roll}}$$

÷ 8

÷ 6

$$B: \frac{\$7.99}{6\text{ rolls}} = \frac{\$1.33}{1\text{ roll}}$$

÷ 6

÷ 2

$$C: \frac{\$3.00}{2\text{ rolls}} = \frac{\$1.50}{1\text{ roll}}$$

÷ 2

**Reduction**

÷ 4

$$A: = \frac{\$8.99}{8\text{ rolls}} = \frac{\$2.25}{2\text{ rolls}}$$

÷ 4

÷ 3

$$B: \frac{\$7.99}{6\text{ rolls}} = \frac{\$2.66}{2\text{ rolls}}$$

÷ 3

$$C: \frac{\$3.00}{2\text{ rolls}}$$

**Common Denominator**

× 3

$$A: = \frac{\$8.99}{8\text{ rolls}} = \frac{\$26.97}{24\text{ rolls}}$$

× 3

× 4

$$B: = \frac{\$7.99}{6\text{ rolls}} = \frac{\$31.96}{24\text{ rolls}}$$

× 4

× 12

$$C: \frac{\$3.00}{2\text{ rolls}} = \frac{\$36.00}{24\text{ rolls}}$$

× 12

factor-of-change strategy (see **fig. 7b**). This strategy was presented after the unit rate strategy to illustrate the simplicity of the computations involved, in contrast to the previous method. Thus, the first comparison task led to a discussion of student-generated unit rate and factor of change strategies and motivated students to consider when one strategy would be more easily applied than another. Additionally, the teacher seized the opportunity to point out that a multiplicative relationship between ratios, as shown in the factor of change strategy, always exists when ratios are proportional. Further, the class discussed how the unit rate strategy is similar to a factor of change strategy. **Figure 8** illustrates how we find the unit price for potatoes at store A by multiplying the provided ratio by a factor of one-tenth, or divide by ten, to get a unit in the denominator.

If someone in our class had used an additive approach to compare these ratios, we would have addressed it by connecting to the context. For instance, if someone had explained that they added 10 pounds to get from 10 pounds to 20 pounds, so they also added $10.00 to the cost to get $11.29, we would have directed the class to notice that this would mean that the first 10 pounds cost $1.29, but the second 10 pounds cost $10.00. Since the cost for the same weight of potatoes should be the same, this additive strategy does not make sense.

Task 2 also involves potatoes; however, in this task one of the provided prices was given as a unit rate. The inclusion of a unit rate further encouraged the use of a unit rate strategy. This task was incorporated to ensure that a unit rate strategy would be

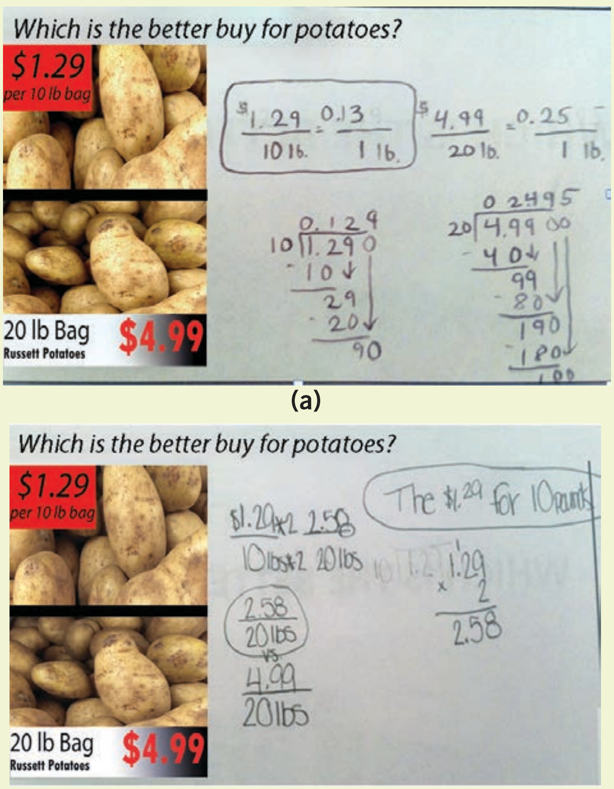
**Fig. 6** This monitoring tool helped us select and sequence who and what would be shared in the whole-class discussion.

| | Strategy | Who and What | Order |
|---|---|---|---|
| Task 1 | Unit Rate | | First |
| | Factor of Change | | Second |
| Task 2 | Unit Rate | | TBD |
| | Factor of Change | | TBD |
| Task 3 | Unit Rate | | First (or omit) |
| | Common Denominator | | Second |
| | Combination | | Third |
| Task 4 | Unit Rate | | First (or omit) |
| | Common Denominator | | Second |
| | Reduction | | Third |
| Task ___ | Other | | |

*Source:* Adapted from Smith et al. (2009)

Note: The cells that are completed in the “order” column specify our anticipated sequence prior to monitoring the classwork. TBD indicates that the order was determined during the monitoring process based on the frequency of that strategy’s use, with the most common strategy shared first.

**Fig. 7** One pair of students used the unit rate strategy on task 1 (a); another used the factor of change strategy for the same task (b).




(a)

(b)

shared. Unlike task 1, we determined the order in which the strategies would be presented while monitoring the classwork. To provide validation, we began with the most commonly used strategy. Again, the class discussed how both strategies, factor of change and unit rate, were related by looking at the multiplicative change involved in each.

In task 3, students compared the prices for differing numbers of 12 packs of soft drinks, \$9.00 for 4 at store A versus \$22.50 for 10 at store B. This numerical structure is unique from the previous two tasks in that it involves equivalent ratios and the factor of change between ratios is not an integer. The aim of this task was to elicit a common denominator strategy (e.g., find the cost of 20 or 40 of the 12 packs at each store) and a combination strategy (e.g., find the cost of 10 of the 12 packs at each store, by finding the cost of 8 and 2 of the 12 packs at store A and combining). **Figure 9** portrays the combination strategy that one group shared.

In the class discussion, we selected two groups who had used the two intended strategies to present their processes for the class. We sequenced the strategies in order of sophistication, with the common denominator strategy presented first. This strategy was deemed less sophisticated because it connected to the students’ prior knowledge regarding fraction equivalence. According to Smith et al. (2009), it can be beneficial to begin with a strategy that is more familiar to students to validate their thinking and allow for connections between prior knowledge (equivalent fractions) and new knowledge (equivalent ratios). The class then discussed the similarities and differences between the common denominator strategy just witnessed and the factor of change strategies seen for task 1 and 2. Together we recognized that the

**Fig. 8** The teacher connected the factor of change strategy for finding an equivalent ratio to the unit rate strategy for finding an equivalent unit rate by illustrating that the unit rate strategy involves multiplying by a fractional factor of change.

Factor of Change

$$\frac{\$1.29}{10} \overset{\times 2}{\underset{\times 2}{=}} \frac{\$2.58}{20}$$

Unit Rate

$$\frac{\$1.29}{10} \overset{\times \frac{1}{10}}{\underset{\times \frac{1}{10}}{=}} \frac{\$0.13}{1}$$

**Fig. 9** A combination strategy involving factor of change and reduction strategies was shared by one group of students.

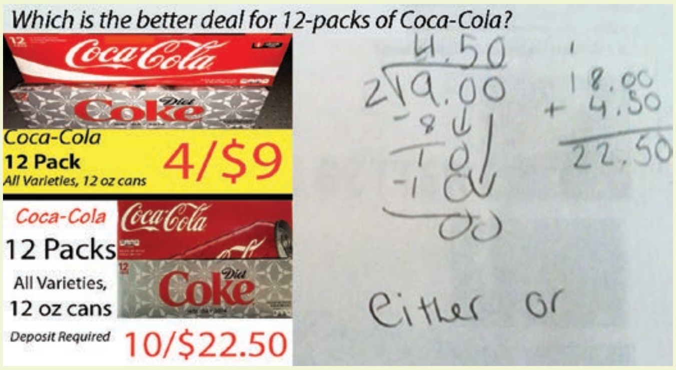



**Fig. 10** The teacher elaborated on the student-generated strategy shared in **figure 5**.

Step 1:
Factor of Change Strategy:

$$\frac{\$9.00}{4\text{ lb.}} \overset{\times 2}{\underset{\times 2}{=}} \frac{x}{8\text{ lb.}},\ x = \$18.00 \Rightarrow \$18.00 \text{ for 8 lb.}$$

Step 2:
Reduction Strategy:

$$\frac{\$9.00}{4\text{ lb.}} \overset{\div 2}{\underset{\div 2}{=}} \frac{y}{2\text{ lb.}},\ y = \$4.50 \Rightarrow \$4.50 \text{ for 2 lb.}$$

Step 3: Combine:

$$8\text{ lb.} + 2\text{ lb.} = 10\text{ lb.}$$
$$\$18.00 + \$4.50 = \$22.50 \Rightarrow \$22.50 \text{ for 10 lb.}$$

common denominator strategy is a factor of change strategy where the common denominator found is not equal to either of the original denominators.

Second, we asked a group of students to explain their combination strategy, which was discovered by few students in the class. We then explicitly connected the students' work to the factor of change and reduction strategies discussed earlier by labeling each step according to the strategy it matched and labeling the multiplicative relationships with arrow diagrams, as shown in **figure 10**. We represented this combination strategy again, but more concretely, in a table (see **fig. 11**). Within the table, we used arrows to mark the multiplicative relationships. We chose to label the reduction from 4 to 2 as division by 2, as opposed to multiplication by 1/2, because our students are more comfortable operating with whole numbers; however, we asked the class what our factor of change would be if we were to think of it as multiplication instead of division, to reiterate that a factor of change always exists.

The final task asked students to compare three different deals for paper towels: 8 rolls for $8.99, 6 rolls for $7.99, and 2 rolls for $3.00. This task appeared last because there were three ratios to compare. The majority of our students used a unit rate strategy to compare the three ratios and, hence, the unit rate strategy was shared first. Next, when monitoring the groups as they worked, we noticed a reduction strategy, determining the price for 2 rolls according to each deal, and a common denominator strategy, finding the price for 24 rolls using each deal. Those groups were asked to detail their approaches for the class. We again connected these approaches to the ones shared earlier in the discussion by depicting, with arrows, the factor of change for each.

**Fig. 11** This model was drawn on the board to further clarify the combination strategy shared in **figure 10** and to illustrate the connection to the previously presented strategies.

| Number of 12 Packs | Cost ($) for 4 12-packs for $9.00 |
|---|---|
| ÷ 2 2 | 4.50 ÷ 2 |
| 4 | 9.00 |
| × 2 8 | 18.00 × 2 |
| 10 (2 + 8 = 10) | 22.50 (4.50 + 18.00 = 22.50) |

We also asked, "Could we have used a factor of change strategy to find the price for 10 rolls?" Students then realized that the factor of change was 2.5, which we related to the combination strategy in which we found the price for 2 groups of 4 rolls and for 1/2 group of 4 rolls. We reiterated that the strategies that we discussed (factor of change, unit rate, common denominator, and combination) were all related to the factor of change strategy because equal ratios always have a multiplicative relationship that can be represented with an arrow diagram. Using the Five Practices and our well-thought-out tasks enabled us to effectively facilitate this student-centered lesson while achieving our content goals.

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## THE END RESULT: QUANTITATIVE REASONING

The Better Buy lesson not only provided an interesting and real-life context for studying proportional reasoning strategies but also required students to reason quantitatively and model with mathematics, two of the mathematical practices delineated within the Common Core State Standards for Mathematics (CCSSI 2010, pp. 6–8). Although this activity was used with an eighth-grade class to review and highlight the multiplicative structure of proportional situations, it is best suited for sixth-grade and seventh-grade audiences before cross multiplication and other proportional reasoning strategies are formally introduced. The students were so engaged in this activity that many groups finished the four assigned tasks and continued on to complete other grocery price comparisons.

Using the Five Practices model during the planning and implementation of this lesson in the classroom, we were able to effectively highlight multiple proportional reasoning strategies and their multiplicative properties while maintaining the student-centered aspect of our instruction. Allowing the students to generate their own methods for comparing the ratios based on their prior knowledge and intuitions enabled us to connect the formal ideas to their informal ones and, in turn, will lead to deeper understandings (de la Torre et al. 2013) of proportionality.

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CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

# Using Proportional Reasoning

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

# Using Proportional Reasoning

## MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to reason proportionally when comparing the relationship between two quantities expressed as unit rates and/or part-to-part ratios. In particular, it will help you assess how well students are able to:

- Describe a ratio relationship between two quantities.
- Compare ratios expressed in different ways.
- Use proportional reasoning to solve a real-world problem.

## COMMON CORE STATE STANDARDS

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

6.RP: Understand ratio concepts and use ratio reasoning to solve problems.

This lesson also relates to **all** the *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 1, 2, 3, 4, and 6:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

## INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding and difficulties. You then review their solutions and create questions for students to consider, in order to improve their work.
- After a whole-class introduction, students work in groups, putting diagrams and descriptions of orange and soda mixtures into strength order. Students then compare their work with their peers.
- Next, in a whole-class discussion, students critique some sample work stating reasons why two mixtures would or wouldn't taste the same. Students then revise and correct any misplaced cards.
- After a final whole-class discussion, students work individually either on a new assessment task, or return to the original task and try to improve their responses.

## MATERIALS REQUIRED

- Each student will need a mini-whiteboard, pen, and eraser, and a copy of *Mixing Drinks* and *Mixing Drinks (revisited)*.
- Each small group of students will need the cut-up *Card Set: Orange and Soda Mixtures* and *Card Set: Blank Cards*, a sheet of poster paper and a glue stick.
- You may wish to have some orange juice and soda for mixing/tasting but this is not essential.

## TIME NEEDED

15 minutes before the lesson, a 100-minute lesson (or two 55-minute lessons), and 15 minutes in a follow-up lesson. Timings given are approximate and will depend on the needs of your class.

## BEFORE THE LESSON

### Assessment task: *Mixing Drinks* (15 minutes)

Have students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the lesson that follows.

**Mixing Drinks**

When Sam and his friends get together, Sam makes a fizzy orange drink by mixing orange juice with soda.

On Friday, Sam makes 7 liters of fizzy orange by mixing 3 liters of orange juice with 4 liters of soda.

On Saturday, Sam makes 9 liters of fizzy orange by mixing 4 liters of orange juice with 5 liters of soda.

1. Does the fizzy orange on Saturday taste the same as or different to Friday's fizzy orange?

   If you think it tastes the same, explain how you can tell.
   If you think it tastes different, does it taste more or less orangey? Explain how you know.

2. On Sunday, Sam wants to make 5 liters of fizzy orange that tastes *slightly less orangey* than Friday's and Saturday's fizzy orange. For every liter of orange, how many liters of soda should be added to the mixture? Explain your reasoning.

Give each student a copy of *Mixing Drinks*. Introduce the task briefly, helping the class to understand the task:

> *This task is about making a fizzy orange drink by mixing different quantities of orange and soda.*
>
> *You are going to compare how orangey the drinks will taste, as well as working out the amount of orange and soda needed to make fizzy orange with a similar orangey taste.*

It is important that, as far as possible, students answer the questions on the sheet without assistance. If students are struggling to get started, ask questions that help them understand what they are being asked to do, but do not do the task for them.

Students should not worry too much if they cannot understand or do everything, because there will be a lesson related to this, which should help them. Explain to students that by the end of the next lesson they should expect to answer questions such as these confidently; this is their goal.

### Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research suggests that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics. Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We recommend that you:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, and highlight the questions for each individual student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students and write these questions on the board when you return the work to the students in the follow-up lesson.

| Common issues | Suggested questions and prompts |
|---|---|
| **Reasons additively rather than multiplicatively**<br>For example: The student states that the fizzy orange tastes the same on Saturday as it did on Friday because one more liter of orange and one more liter of soda has been added and these just 'cancel each other out' (Q1).<br>Or: The student states that the fizzy orange tastes the same on Saturday as it did on Friday because both mixtures contain one more liter of soda than orange (Q1). | • How could you use math to check that the addition of a liter of orange and a liter of soda has no effect on the taste?<br>• What would happen to the taste if a liter of orange and a liter of soda were added to 1 liter of soda?<br>• If 3 liters of fizzy orange was made in the same way, by mixing 1 liter of orange with 2 liters of soda, would this taste the same also? |
| **Sole focus on orange as the 'active' ingredient**<br>For example: The student thinks that Saturday's fizzy orange will taste more orangey than Friday's, because it has more orange in it than Friday's has (Q1). | • How much soda is in Saturday's fizzy orange? How much soda is in Friday's fizzy orange? What do you notice?<br>• Is how orangey the fizzy orange tastes determined by the number of liters of orange it contains? |
| **Sole focus on soda as the diluting ingredient**<br>For example: The student thinks that Saturday's fizzy orange will taste less orangey than Friday's, because it has more soda in it than Friday's has (Q1). | • How much orange is in Saturday's fizzy orange? How much orange is in Friday's fizzy orange? What do you notice?<br>• If 5 liters of fizzy orange were made by mixing 4 liters of soda with 1 liter of orange, would it also taste more orangey than Saturday's fizzy orange? |
| **Provides an explanation based on one mixture only**<br>For example: The student states that Saturday's fizzy orange will taste less orangey than Friday's, because the mixture contains less orange in it than soda (Q1). | • Does Friday's fizzy orange contain more orange than soda or more soda than orange?<br>• How can you compare the taste of Saturday's fizzy orange to the taste of Friday's fizzy orange? |
| **Makes incorrect assumptions**<br>For example: The student thinks that on Sunday, Sam should mix 1 liter of orange with 4 liters of soda because 2 liters of orange with 3 liters of soda will taste the same as Friday's and Saturday's fizzy orange (Q2).<br>Or: The student assumes that for every liter of orange two liters of soda are required (Q2). | • Will this fizzy orange mixture taste *slightly* less orangey than Friday's and Saturday's fizzy orange? |
| **Provides little mathematical explanation** | • Can you use math to explain your answer? |
| **Completes the task correctly**<br>The student needs an extension task. | • Can you find a fizzy orange mixture that is more orangey than Friday's fizzy orange but less orangey than Saturday's fizzy orange? |

## SUGGESTED LESSON OUTLINE

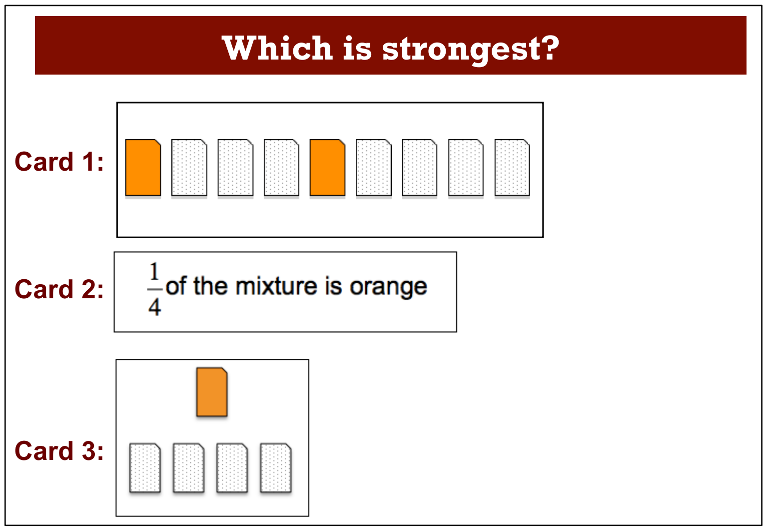
### Whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen, and eraser. Remind the class of the assessment task they have already attempted.

*Recall what we were working on previously. What was the task about?*

*In today's lesson we are going to consider different mixtures of orange and soda used to make fizzy orange and think about which ones taste more/less orangey.*

Display Slide P-1 of the projector resource:




*Each of these three cards describes a fizzy orange mixture.*

*The diagrams on cards 1 and 3 show the amount of orange and soda in the mix (where the shaded boxes represent the orange and the dotted boxes represent the soda) and card 2 gives a description of the fraction of the fizzy orange mixture that is orange.*

*Working on your own, on your mini-whiteboard, write the card numbers in order from least orangey to most orangey. [Card 3, Card 1, Card 2.]*

Give students a few minutes to work on this before asking to see their whiteboards. If there are a range of responses within the class, collate them on the board and hold a whole-class discussion. Spend a few minutes discussing the strategies used to compare the three cards.

Explain to students that they are going to be working in groups on a similar activity putting cards in order of strength from least orangey to most orangey.

### Individual think time, then collaborative work: *Orange and Soda Mixtures* (30 minutes)

Before students work collaboratively, it can be helpful to give students individual 'thinking time'. This allows everyone to have time to construct ideas to share and avoids the conversation being dominated by one student.

Organize students into groups of two or three. Give each group the cut-up *Card Set: Orange and Soda Mixtures*, a sheet of poster paper, and a glue stick.

*On these cards there are descriptions of fizzy orange mixtures.*
*Some cards show the number of orange and soda juice boxes in the mixture, some contain a written description of the mixture and some show empty juice boxes which you will need to shade in (color orange juice boxes and draw dots for soda.)*

Display Slide P-2 of the projector resource:

**Individual think time**

Your task is to work with your partner to put the cards in order of strength, from least orangey (on the left) to most orangey (on the right).

1. Look at the cards and think about ways you could carry out this task.

2. Write your ideas on your mini-whiteboards.

There is no need for students to order the cards during this individual activity.

When students have had sufficient time to think about the task:

*First, take turns to explain to each other your ideas for how to carry out the task.*

*Ask questions if you do not understand your partner's explanation.*

*Take a few minutes to come up with a joint plan of action.*

Display Slide P-3 of the projector resource and explain how students are to work together on the task:

**Working together**

1. Work together to put the cards in order of strength, taking turns with the work.
   a. Explain decisions to your partner.

2. If you think more than one card describes the same fizzy orange mixture, group them together.
   a. If a group of cards does not contain a juice box card, then shade in one of the Cards M - P.

3. When you both agree where each card should go and why, glue them onto your poster. On your poster, explain your decisions.

While students are working, you have two tasks: to notice their approaches to the task and to support student problem solving.

### Make a note of student approaches to the task

Listen and watch students carefully. In particular, notice how students make a start on the task, where they get stuck, and how they overcome any difficulties.

Do they begin with what they think is the strongest or weakest mixture or do they just pick a random card? Do students compare orange to soda (e.g. for every orange there are 2 soda) or orange to mixture (e.g. ½ the mixture is orange). When they discover cards that are of equal strength, how do they justify this to one another? Do they use fractions, decimals, percentages, ratios or proportions? Do they switch between different descriptions? How do they go about shading cards M to P?

You can use this information to focus a whole-class discussion towards the end of the lesson.

### Support student problem solving

As students work on the task support them in working together. Encourage them to take turns and if you notice that one partner is doing all the ordering or that they are not working collaboratively on the task, ask a student in the group to explain a card placed by someone else in the group.

Try not to make suggestions that push students towards a particular approach to the task. Instead, ask questions to help students clarify their thinking. The following questions and prompts may be helpful:

*Which mixture do you think is the most orangey? Why?*

*How do you know that this mixture is more orangey than that one?*

*Why does this card come here?*

Encourage students to write on the cards.

If several students in the class are struggling with the same issue, you could write one or two relevant questions on the board and hold a brief whole-class discussion. For example, if students are using 'additive' rather than 'multiplicative' reasoning; e.g. thinking that 3:5 (Card B) is the same as 4:6 (Card E) you could ask:

*Why do you think that these will taste the same?*

*Can you think of another fizzy orange mixture that will also taste the same? How do you know?*

Students who finish early with the cards in the right order could be given cut-up *Card Set: Blank Cards* and asked:

*Can you invent a card that would go in between these two?*

*Can you invent a card that would go in the same place as this one?*

*What would you add to this mixture to make it taste like this mixture?*

## Sharing work (15 minutes)

Give students the opportunity to compare their work by visiting another group. It is likely that some groups will not have ordered all the cards but a comparison can still be made as to whether students consider a particular card to be more orangey or less orangey than another. It may be helpful for students to jot down on their mini-whiteboards their agreed order of the cards before they visit another group.

Show Slide P-4 and explain how students are to share their work:

**Sharing work**

1. One person from each group get up and visit a different group.
2. If you are staying with your poster, explain your card order to the visitor, justifying the placement of each card.
3. If you are the visitor, look carefully at the work and challenge any cards that you think are in the wrong place.
4. If you agree on the placement of the cards, compare your methods used when ordering.

### Extending the lesson over two days

If you are taking two days to complete the unit then you may want to end the first lesson here. At the start of the second day, allow time for students to remind themselves of their work before moving on to discuss their ordering of the cards as a whole-class.

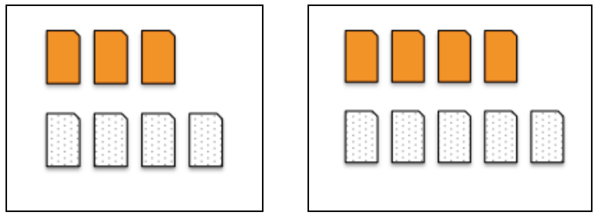
## Whole-class discussion (25 minutes)

Now hold a brief whole-class discussion in which students discuss their ordering. Draw attention to significant differences between the ordering that particular groups have arrived at.

*Were there any disagreements when you compared your work? Someone give me an example. What reasoning did you each give?*

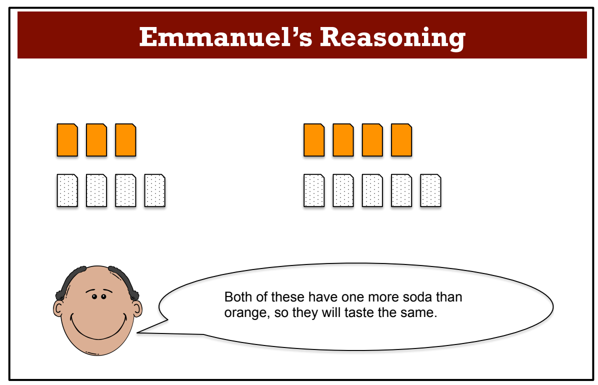
*Was different math used to figure out the ordering? [E.g. orange to soda or orange to mixture]*

Once you have had a chance to compare reasons given, spend some time exploring conflicting reasoning/conclusions when comparing the following two fizzy orange mixtures:



Display Slide P-5 of the projector resource to show Emmanuel's reasoning and ask:

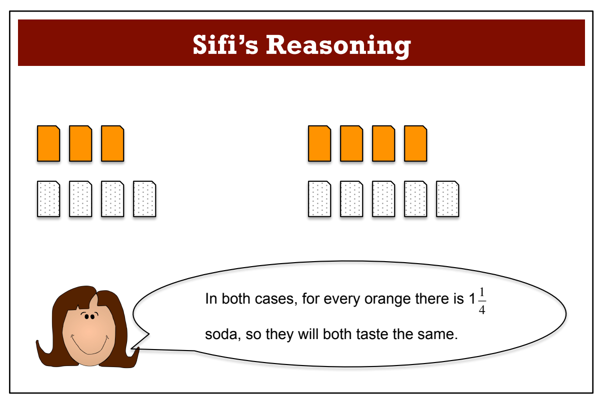
*What do you think about Emmanuel's reasoning? Is he right or wrong? Why?*




Students should be suspicious of this kind of 'linear' reasoning by now and if they are not you could explore what would happen if you continued the pattern to the left (2 orange and 3 sodas, 1 orange and 2 sodas etc.). Taking one more step to the left we would have no orange and 1 soda. There is still 'one more soda than orange' but everyone will agree that this will not taste orangey at all!

Now display Slide P-6 of the projector resource showing Sifi's reasoning and ask:

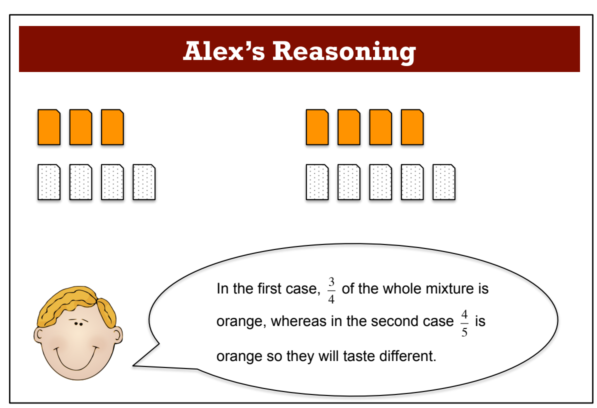
*What do you think about Sifi's reasoning? Is she right or wrong? Why?*




Sifi's method is better than Emmanuel's because she is thinking proportionately, but she has made an error; $1\frac{1}{4}$ is correct for the right-hand mixture, for the number of soda juice boxes per orange juice box, but the left-hand mixture is $1\frac{1}{3}$.

Now use Slide P-7 of the projector resource to display Alex's reasoning and ask:

*What do you think about Alex's reasoning? Is he right or wrong? Why?*




Alex has come to the correct conclusion about the mixtures not tasting the same but his method contains an error. The left-hand mixture is $\frac{3}{7}$ orange not $\frac{3}{4}$ orange (it is in the ratio of 3:4 (orange:soda)) and the right-hand mixture is $\frac{4}{9}$ orange not $\frac{4}{5}$ (ratio 4:5). Since $\frac{3}{7}$ is less than $\frac{4}{9}$, the right-hand mixture will be *slightly* more orangey (but it may be hard to tell this small difference in practice!)

Finally, you might want to ask:

*Did you use any of these methods? Which ones?*

*Did you use any other methods? What were they?*

*What do you think now about all of these methods?*

### Poster review (10 minutes)

Students now have an opportunity to reconsider the ordering of their cards:

*Now that you have had a chance to compare and discuss your work and we have looked at what Emmanuel, Sifi and Alex have said, you might like to have another look at your poster and decide in your groups whether you are still happy with where you have placed the cards.*
*If you think a card is in the wrong place, draw an arrow on your poster to where you think it should go.*

While this is happening, encourage students to voice their reasoning for the movement of a card.

### Whole-class discussion (10 minutes)

You may want to finish with a brief whole-class discussion in which students discuss their ordering and talk more generally about what they have gained from the lesson.

*Did you change your ordering after we talked together about it? Why / Why not?*

*How confident are you with your ordering now?*

*What have you learnt today about how you get mixtures that taste the same or different?*

Use your knowledge of the students' group work to call on a wide range of students for contributions.

### Follow-up lesson: reviewing the assessment task (15 minutes)

Give students their responses to the original assessment task *Mixing Drinks* and a copy of the task *Mixing Drinks (revisited)*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students then select from this list only those questions they think are appropriate to their own work.

*Look at your original responses and the questions [on the board/written on your paper]. Answer these questions and revise your response.*

*On your mini-whiteboard make some notes on what you have learned during the lesson. Now have a go at the second sheet: Mixing Drinks (revisited). Can you use what you have learned to answer these questions?*

If students struggled with the original assessment task, you may feel it more appropriate for them to revisit *Mixing Drinks* rather than attempting *Mixing Drinks (revisited)*. If this is the case give them another copy of the original assessment task instead.

If you are short of time you could give this task for homework.

## SOLUTIONS

### Assessment task: *Mixing Drinks*

1. The ratio of orange to soda on Friday is 3:4, which is not equal to the ratio of orange to soda on Saturday (4:5), so the fizzy orange mixtures will not taste the same. Friday's mixture is $\frac{3}{7}$ orange and Saturday's mixture is $\frac{4}{9}$ orange. Comparing these fractions to see which will taste the most orangey ($\frac{3}{7} = \frac{27}{63}$ compared with $\frac{4}{9} = \frac{28}{63}$) reveals that Saturday's fizzy orange mixture will taste more orangey. However, students may comment that even though Saturday's fizzy orange is stronger than Friday's, it is likely that you would not be able to taste any difference because the difference is only very slight.
2. If Sam mixes 2 liters of orange with 3 liters of soda, the mixture will be $\frac{2}{5}$ orange, which is *slightly* less orangey than Friday's and Saturday's mixture. This means that for every liter of orange, $1\frac{1}{2}$ liters of soda should be added to the mixture.

### Assessment task: *Mixing Drinks (revisited)*

1. The completed table is as follows: (missing values are identified in **bold**)

| Amount of Raspberry Juice (liters) | Amount of Apple Juice (liters) | Amount of Soda (liters) | Total Amount of *Fabulous Fruit Fizz* (liters) |
|---|---|---|---|
| 1 | 2 | 3 | 6 |
| **0.5** | 1 | **1.5** | **3** |
| **2** | **4** | **6** | 12 |

2. a. $\frac{2}{5}$ of the drink is apple juice.

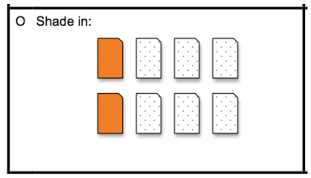
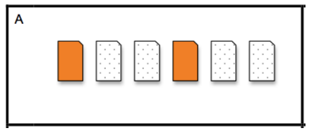
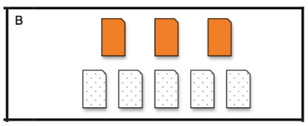
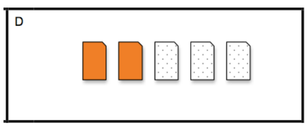
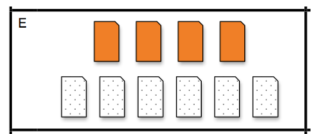
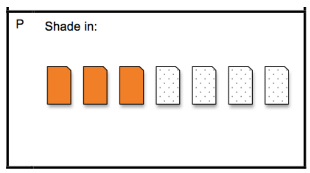
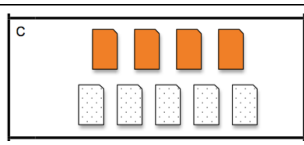
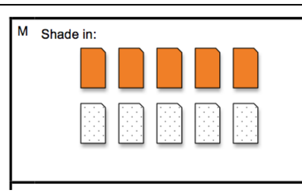
   b. $\frac{2}{5}$ of the drink is apple juice.

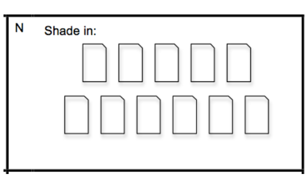
   c. $\frac{1}{3}$ of the drink is apple juice.

   Mixture c is the least appley drink. Qaylah should mix, for every liter of apple, 2 liters of soda.

## Collaborative task:

The correct matching/ordering from least orangey to most orangey (with ratio of orange to soda also given) is as follows:

| | | | |
|---|---|---|---|
| 1:3 | I<br>**One fourth of the mixture is orange** |  | |
| 1:2 | J<br>**$\frac{2}{3}$ of the mixture is soda** |  | G<br>**For every orange there are 2 sodas** |
| 3:5 |  | | |
| 2:3 |  | L<br>**For every soda there is $\frac{2}{3}$ orange** |  |
| 3:4 | K<br>**For every orange there is $1\frac{1}{3}$ soda** |  | |
| 4:5 |  | H<br>**Orange : Soda = 4 : 5** | |
| 1:1 | F<br>**Half of the mixture is orange** |  | |




Card N has been designed so that it cannot be shaded to be equivalent to any of the other cards. Students should shade the card with a number of orange/soda juice boxes of their choice (between 1 and 10) and then place it in the appropriate place based on how orangey the mixture is.

For example, they may choose to shade it in the ratio of 5 orange: 6 soda and place it between cards C and M.

# Mixing Drinks

When Sam and his friends get together, Sam makes a fizzy orange drink by mixing orange juice with soda.

On Friday, Sam makes 7 liters of fizzy orange by mixing 3 liters of orange juice with 4 liters of soda.

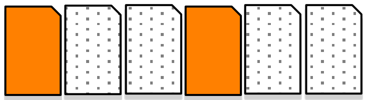
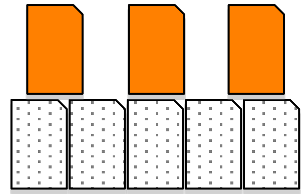
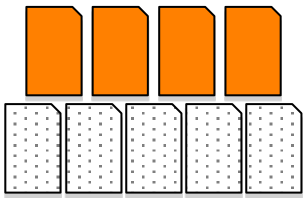
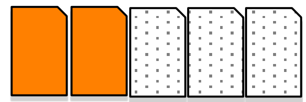
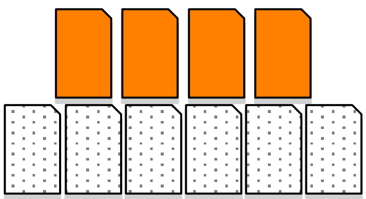
On Saturday, Sam makes 9 liters of fizzy orange by mixing 4 liters of orange juice with 5 liters of soda.

1. Does the fizzy orange on Saturday taste the same as Friday's fizzy orange, or different?

   If you think it tastes the same, explain how you can tell.
   If you think it tastes different, does it taste more or less orangey? Explain how you know.

2. On Sunday, Sam wants to make 5 liters of fizzy orange that tastes *slightly* less orangey than Friday's and Saturday's fizzy orange. For every liter of orange, how many liters of soda should be added to the mixture? Explain your reasoning.

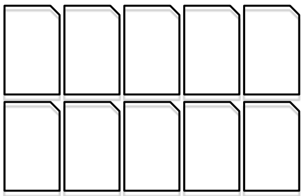
# Card Set: Orange and Soda Mixtures

| | |
|---|---|
| A  | B  |
| C  | D  |
| E  | F **Half of the mixture is orange** |
| G **For every orange there are 2 sodas** | H **Orange : Soda = 4 : 5** |
| I **One fourth of the mixture is orange** | J $\frac{2}{3}$ **of the mixture is soda** |
| K **For every orange there is** $1\frac{1}{3}$ **soda** | L **For every soda there is** $\frac{2}{3}$ **orange** |

# Card Set: Orange and Soda Mixtures (continued)

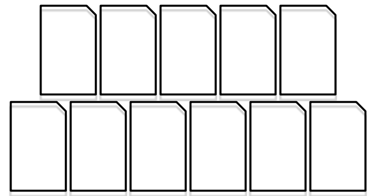
M

Shade in:

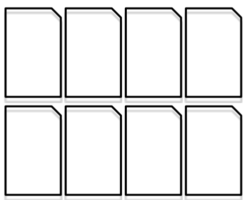


N

Shade in:

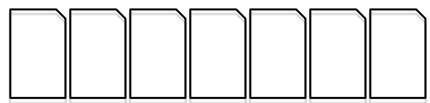


O Shade in:



P

Shade in:



# Card Set: Blank Cards

# Mixing Drinks (revisited)

To make 6 liters of *Fruit Fizz*,
mix 1 liter of raspberry juice, 2 liters of
apple juice and 3 liters of soda

1. Complete the table below with the amounts of raspberry juice, apple juice and soda needed to make the different quantities of *Fruit Fizz*. The mixture must taste exactly the same each time.

| Amount of Raspberry Juice (liters) | Amount of Apple Juice (liters) | Amount of Soda (liters) | Total Amount of *Fruit Fizz* (liters) |
|---|---|---|---|
| 1 | 2 | 3 | 6 |
| | 1 | | |
| | | | 12 |

2. Here are three ways to make apple fizz:

   a. For each liter of soda mix $\frac{2}{3}$ liters of apple juice.

   b. Mix apple and soda in the ratio 2 : 3.

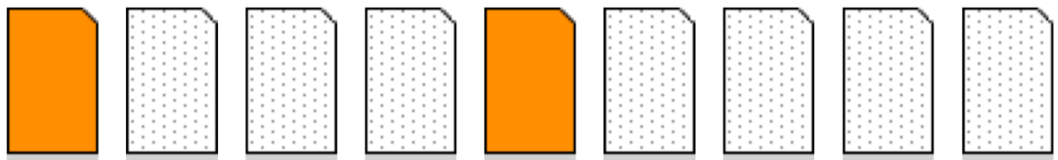
   c. $\frac{2}{3}$ of the mixture is soda, the rest is apple juice.

Qaylah wants to mix the least appley drink. Which mixture should she choose?

For every liter of apple, how many liters of soda should she add to the mixture?
Explain your reasoning.

# Which is strongest?

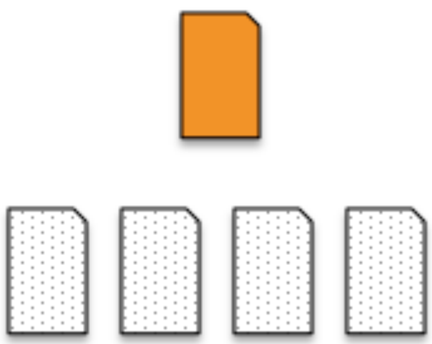
**Card 1:**



**Card 2:**

$\frac{1}{4}$ of the mixture is orange

**Card 3:**



# Individual think time

Your task is to work with your partner to put the cards in order of strength, from least orangey (on the left) to most orangey (on the right).

1. Look at the cards and think about ways you could carry out this task.

2. Write your ideas on your mini-whiteboards.

# Working together

1. Work together to put the cards in order of strength, taking turns with the work.
   a. Explain decisions to your partner.

2. If you think more than one card describes the same fizzy orange mixture, group them together.
   a. If a group of cards does not contain a juice box card, then shade in one of the Cards M - P.

3. When you both agree where each card should go and why, glue them onto your poster. On your poster, explain your decisions.

# Sharing work

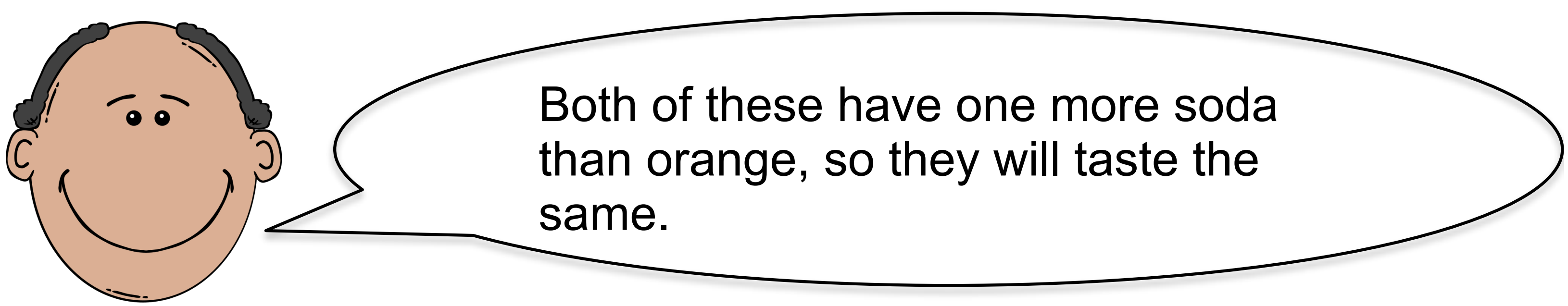
1. One person from each group get up and visit a different group.

2. If you are staying with your poster, explain your card order to the visitor, justifying the placement of each card.

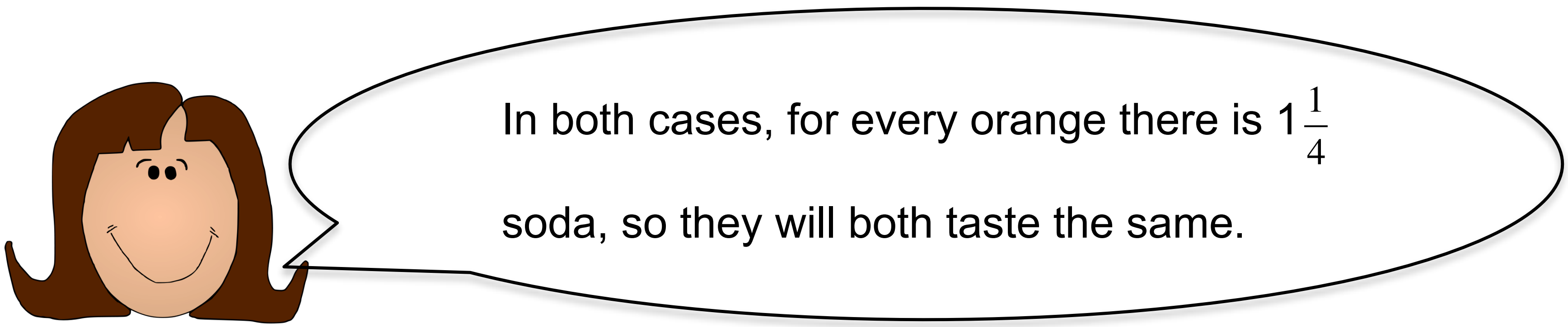
3. If you are the visitor, look carefully at the work and challenge any cards that you think are in the wrong place.

4. If you agree on the placement of the cards, compare your methods used when ordering.

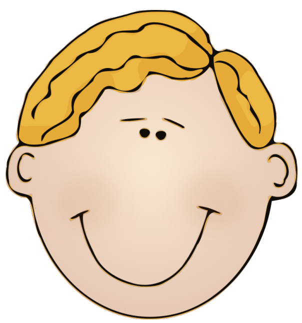
# Emmanuel's Reasoning



# Sifi's Reasoning



# Alex's Reasoning



In the first case, $\frac{3}{4}$ of the whole mixture is orange, whereas in the second case $\frac{4}{5}$ is orange so they will taste different.

Mathematics Assessment Project

# Classroom Challenges

These materials were designed and developed by the
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The classroom observation teams in the US were led by
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This project was conceived and directed for
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The full collection of Mathematics Assessment Project materials is available from

http://map.mathshell.org

# KITCHEN GARDENS

## CONTEXTS FOR DEVELOPING PROPORTIONAL REASONING

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It is great to see how the sharing of ideas sparks new ideas. In 2011 Lyon and Bragg wrote an APMC article on the mathematics of kitchen gardens. In this article the authors show how the kitchen garden may be used as a starting point for proportional reasoning. The authors highlight different types of proportion problems and how the authentic context of a kitchen garden may be used to spark interest in reasoning.

### Introduction

Across Australia, many schools have kitchen gardens. Some of these schools have been developed through the Stephanie Alexander Foundation while others, including the school described here, have chosen to create their own kitchen garden with the help of the school community. Lyon and Bragg (2011) described ways to integrate mathematics with other curriculum areas through the creation of a kitchen garden. This article focuses on activities used to engage students in a variety of mathematical situations involving proportional reasoning through a series of lessons in their school's kitchen garden. It also identifies the proportional reasoning problem types that arose through the activities.

Proportional reasoning is a key component of numeracy. It involves the ability to understand and use multiplicative relationships in situations of comparison (Behr, Harel, Post & Lesh, 1992). The importance of proportional reasoning in primary school children's mathematics education has long been recognised. Lesh, Post and Behr (1988) described it as the capstone of elementary school arithmetic and the cornerstone of the mathematics learning that follows. Being such a pivotal aspect of numeracy, the development of proportional reasoning skills is critical if children are to be well placed to succeed in mathematics beyond primary and indeed middle schooling. Failure to develop proportional reasoning ability by adolescence can also preclude students from participation in subjects beyond the middle years, including science, mathematics, and technology (Lanius & Williams, 2003).

Generally speaking, situations of proportion require some application of multiplicative or relative thinking. A variety of proportional reasoning problem types are identified in the literature. For example, Lamon (1993) identified the following types of proportion problems:

- *rate problems* (involving both commonly used rates, such as speed, and rate situations in which the relationship between quantities is defined within the question);
- *part–part–whole* (e.g., ratio problems in which two complementary parts are compared with each other or the whole); and
- *stretchers and shrinkers* (growth or scale problems).

In addition, according to Lesh et al. (1988), certain problem types are often neglected in textbooks and classroom instruction. These include problems that require transformation of representational types or modes. While providing students with opportunities to engage in a variety of proportional reasoning situations is important, it is equally important to expose students to situations that are non-proportional in nature (Bright, Joyner & Wallis, 2003) because students often rely on proportional reasoning in circumstances that do not require it — e.g., constant, linear and additive situations (Van Dooren, De Bock, Hessels, Janssens & Verschaffel, 2005).

Proportional reasoning is very often used in real-life mathematics; for example, comparing costs at the supermarket or estimating the travel time required to reach a destination on time. In schools, there exist many opportunities to develop students' proportional reasoning skills in authentic contexts. The focus of this article is the rich context of the kitchen garden.

## Enhancing proportional reasoning in context

The authors are leading a project involving 28 schools in Queensland and South Australia. The project aims to enhance proportional reasoning education through a series of workshops with teachers within six school clusters over a period of two years. Each school cluster includes three to five primary schools with at least one of their local secondary schools. The research team works within clusters and individual schools to support teachers to develop activities that promote proportional reasoning across subject areas and within contexts relevant to each school or cluster. The schools in one of the participating clusters have a long history of collaboration and several of them have developed kitchen garden programs, either through the Stephanie Alexander Foundation or independently. Such programs involve students designing and planting gardens, growing and harvesting vegetables and herbs, and using their produce to create meals for themselves and their classmates, and, in some cases, the broader student community.

## Lessons from one school

The research team were invited by one of the schools to work alongside their Year 5 teachers to develop resources and strategies for enhancing students' proportional reasoning through the school's kitchen garden program. In this school, students from each year level work on the project over the course of a school term during weekly 90-minute sessions (over approximately 10 weeks). Each week, one half of the students work on the garden (planting, soil testing, making compost, harvesting, etc.) while the other half of the students work in the kitchen (preparing, cooking and serving lunch). The groups alternate weekly so that over the term, all students will have spent about five sessions in the garden and five in the kitchen.

On the day that we observed the Year 5 kitchen garden class, the gardening students undertook activities that provided numerous opportunities for the teacher to engage the students in proportional reasoning and to foreground examples of proportional and non-proportional situations. These activities included investigating the components of soil samples and pH measurements. To investigate the different components in their soil samples, the students created water slurries in glass jars. This provided a range of proportional situations, including determining the relative amounts of the different components (part–part–whole comparisons) and comparing and identifying the various components according to their relative densities.

The students used pH kits with colour charts to determine the pH of soil samples. This allowed the teacher to draw the students' attention to an example of a non-proportional situation in which the scale appeared to be proportional and to help the students understand why this was not the case. (The pH scale is an example of a non-proportional scale; it is exponential — an increase of 1 on the scale represents a ten-fold decrease in acidity).

The kitchen is another rich source of opportunities to foreground proportional reasoning situations. On the day of our visit, the students were making potato marsala, breads and fruit kebabs. During preparation of the marsala, their discussions with the researchers centred around the size of potato pieces to ensure they cooked in the time available (a rate situation) and the ratio of different ingredients, depending on the number of people to be served (multiplicative thinking). The students were asked questions that involved manipulating the recipes, such as, "If I had three sweet potatoes instead of two, how many potatoes would I need to keep my vegetables in proportion?"

When making the bread dough, each student had to divide his or her dough into 15 pieces. This led the students to discuss the best shape into which to form their dough so that it could be easily divided into equal pieces. The students initially agreed that a circle would be best but once they started trying to break it into 15 pieces, it became evident that perhaps a different shape would be more useful because it was not an easy task. One student suggested a square and after a short time, the students decided as a group that a rectangle would be the best starting shape, as one student pointed out that 15 is not a square number. They then divided the rectangle into thirds, each of which they further divided into fifths. Figure 1 shows photographs of some of the students' 'dough shapes'. The photograph on the right illustrates the way in which the students divided the dough into 15 pieces.

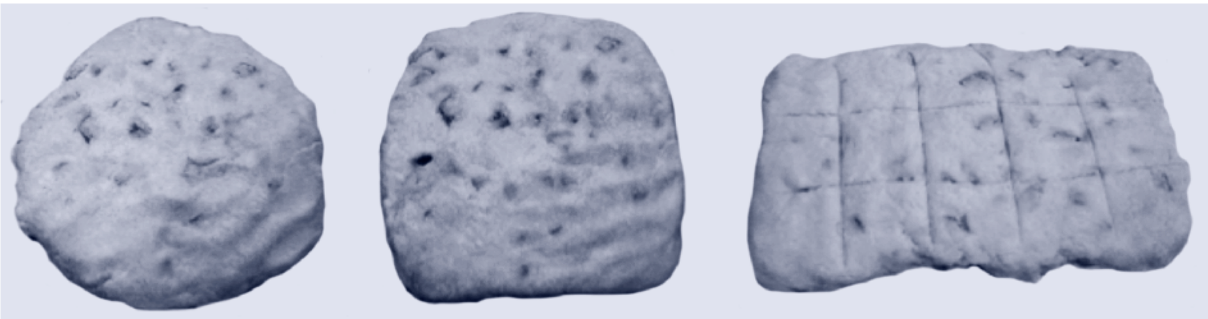

Figure 1. The dough shapes created by students.

This activity led to further discussions of situations in which a circle or a square might be a useful alternative to the rectangle. It provided the students with an opportunity to consider appropriate ways of representing parts and the whole (transforming representations). Such discussions are also valuable for promoting students' understanding of number. For example, the students soon realised that a square number would be most suited to a square shape whereas other numbers could be better represented as an array using the rectangular shape. In the case of the dough, the students created a $3 \times 5$ array. They agreed that the circle was difficult to divide into equal pieces, especially when the required number of pieces was odd.

The task of making fruit kebabs with a variety of five fruits provided opportunities to ask the students further questions about ratio. For example, one of the researchers asked the students to make the kebabs using a particular ratio of fruit: 1 : 2 : 1 : 2 : 1. The students each created a kebab in the required ratio without difficulty. However, when the ratio changed to $1 : \frac{1}{2} : 1 : \frac{1}{2} : 2$, and the students were challenged to create the kebab without cutting any pieces, the situation became more challenging. The students discussed the situation together before a student finally suggested, "doubling all the numbers will give us whole numbers". Once this idea was tabled, the students were able to create the desired kebabs. Again, this activity is an example of a simple situation in which the students were exposed to somewhat challenging ideas but through hands-on activity and group discussion, were able to reach a plausible solution to a part–part–whole problem.

While activities such as these may appear simple at first glance, they allow the students to engage in authentic problem-solving using a variety of ideas, including geometric shapes, arrays and number properties.

## Case study teacher observations

At the beginning of the project, we met with the school principal, the Head of Curriculum, and the coordinating teacher of the kitchen garden. They asked us to develop a series of posters that could be used by the teachers while they were in the kitchen to draw students' attention to situations involving proportional reasoning and to prompt students' thinking. The posters were placed in the kitchen and have been used in mathematics lessons as a stimulus for students' problem solving discussions (see Appendix 1 for examples of the posters). The posters included prompts about the problem types, the types of thinking involved, and opportunities to use important terminology, such as 'relative', 'absolute', 'additive', and 'multiplicative'. This was done firstly to draw the students' attention to the types of thinking in which they were engaging and to encourage them to use the mathematical language. It also provided support for the teachers during their lessons.

After using the posters in class, the kitchen garden coordinating teacher reported in an interview that she had become more aware of the potential of the kitchen garden program for providing opportunities to engage the children in proportional reasoning. She stated that she was more likely to take the time to foreground proportional reasoning and to discuss it with the students. She also reported that in follow-up lessons in the classroom, she observed the students using proportional reasoning without being prompted to solve real problems as they arose in the garden. For example, the students were tasked with planning and building a new garden bed and needed to design an irrigation system. This became a rich numeracy activity, in which the students drew scale diagrams of the garden and superimposed diagrams to investigate the shapes and area of coverage for different garden sprinklers (two-dimensional scale). They also calculated and compared the flow rates from the tap to identify the most water-efficient sprinklers (unfamiliar rate problem). The teacher stated, "They were using proportional reasoning beyond their expected skill levels because they had a real reason for finding out the answers."

## Future plans

In addition to the posters, other resources are being created to support the teachers and parent helpers in the kitchen garden program. Reflecting on the questions we had asked the students during our visit, one of the teachers noted that often teachers were so busy coordinating the students and ensuring that everything ran to time that they missed opportunities to engage the students in proportional thinking. In response, a series of question prompts to which the parents and teachers could refer during the kitchen garden sessions were devised. An example is shown in Appendix 2. It is envisaged that such resources will be devised to accompany each kitchen activity.

In most cases, one of the teachers makes the decisions regarding the amount of each ingredient that is required, based on the number of lunch orders received. There are plans in the future to engage the students more in these decisions, such as using the numbers of servings to determine the required multiple of each of the recipes, as well as assisting with decisions about quantities of ingredients to be ordered.

## Benefits beyond kitchen garden programs

Not all schools have gardens or the resources to allow the students to carry out food preparation. The ideas described in this article grew from one school's kitchen garden project. Through sharing them with other teachers involved in the project, some teachers have been prompted to start a vegetable garden with their class. Teachers without such facilities have still found the activities and resources useful because they focus on authentic, everyday activities in which the students may engage in their lives beyond school. Teachers have used the posters in a variety of ways, sometimes as a stimulus for discussion, at other times, as a means of introducing a new topic or concept. Other teachers have used them for 'problem of the week' ideas. One teacher used the poster shown in Appendix 1 to introduce a mathematical investigation into scale factor and the effect on the volume of objects when one enlarges the shape in one, two or three dimensions.

When seeking to develop students' proportional reasoning skills, it is important to foreground situations of proportion and to engage students in proportional thinking in a variety of contexts. This article has described one approach to engaging students in such ideas through the context of kitchen gardens. The ideas started as a means of supporting the teachers in one school to engage students in a specific program. It has become clear to us that such ideas can be adapted and used effectively in a range of setting, across a number of year levels and for different purposes, thanks to the creativity and professionalism of the teachers involved.

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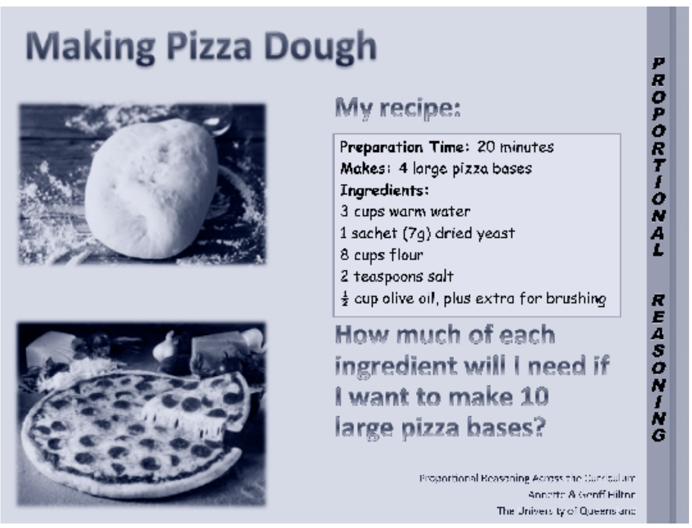
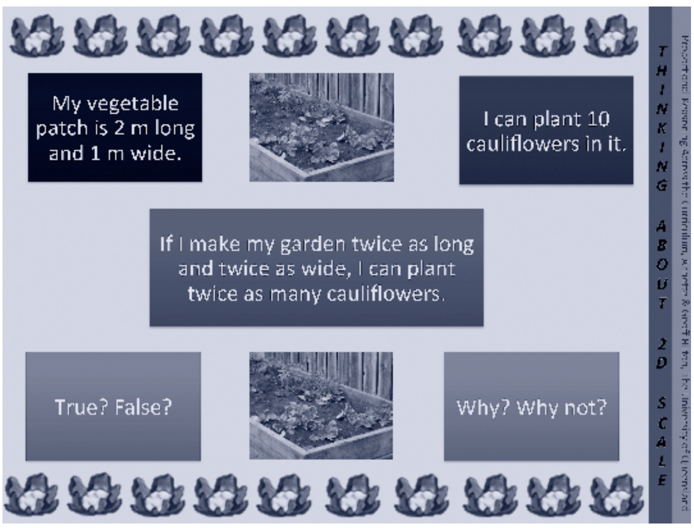
## Appendix

### Appendix 1. Examples of kitchen garden posters



## Appendix 2. Guide for kitchen garden helpers

**Sweet Potato Marsala**

**INGREDIENTS**

8 tablespoons of oil
8 teaspoons of black mustard seeds
2 teaspoons of turmeric
8 cm piece of ginger: grated
2 onions
4 sweet potatoes
16 potatoes
spinach
4 cups of water to cook potatoes

Below are some possible questions to engage student proportional reasoning while preparing the recipe. This recipe example has numbers/amounts that are reasonably easy for students to manipulate, as they are all multiples of two. Many recipes may have numbers/amounts that will require thoughtful questioning to suit the mathematical understandings of the students. Also note that often a similar question can be asked in different ways.

1. *Additive/multiplicative:* If I made the recipe with three onions, to keep everything in proportion, how many sweet potatoes would I need?
   a. Possible responses: Some students may think additively, that is, they may think they have one more onion so they need one more sweet potato. We want them to think multiplicatively, that is, they have 50% more onions or half as many again so they need 50% more sweet potatoes, i.e., 6 sweet potatoes.
2. *Additive/multiplicative:* If I only had one onion, to keep the recipe in proportion, how many potatoes would I need?
   a. Possible response. This is similar to the first question but is a reduction. Again some students may think additively, i.e., reduce the onions by one and therefore the potatoes by one to a total of 15. Thinking multiplicatively, a student would note the number of onions have been reduced by 50% (or halved) so the potatoes would also need to be reduced by 50% (or halved) to 8 potatoes.
3. *Proportional/non-proportional:* Which ingredient listed is not strictly proportional to the other ingredients?
   a. The amount of water to cook the potatoes, while it could be varied with the number of potatoes to be cooked, does not need to be adjusted proportionally for success with the recipe.
   b. Spinach does not have an amount so must be added at the discretion of the chef/cook and therefore not strictly proportional.
4. *Proportions involving fractions:* If I only had 12 potatoes, to keep the recipe in proportion, how many onions would I need?
   a. This is a more difficult question as it involves the students' fractional thinking. Twelve potatoes are $\frac{3}{4}$ (or 75%) of sixteen potatoes so the recipe would need $\frac{3}{4}$ (or 75%) of two onions, i.e., $1\frac{1}{2}$ onions.
   b. This question can cause confusion with children who are additive thinkers. They may think they have four fewer potatoes and need four fewer onions but only have two. This could be a good way of demonstrating that when thinking proportionally, additive thinking does not work.

# Mixing Paints With Ratios (6G)

**Topic: Equivalent Ratios**

**Learning goals**

Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.
Use ratio and rate reasoning to solve real-world and mathematical problems.

Sim
Proportion Playground

**Explore**

1. Play with Proportion Playground.

**Generate Cases**

2. Create a hue of orange. Then try to create the same orange, with different amounts of red and yellow paint. Write down the ratios that make the same hue.

3. Compare your work with someone else's.

**Conjecture**

4. Make a conjecture about what is always true about the ratios that produce the same color orange.

**Justify**

5. Justify using what you know about paint, numbers, symbols, or diagrams.

**Conclude**

6. Write down the conclusion you think is most important.

**Extension**

7. A painter tries the sim and says "10 reds and 11 yellows make the same hue of orange as 11 reds and 12 yellows, so 10:11 = 11:12". Do you agree or disagree? Give a justification for your answer.

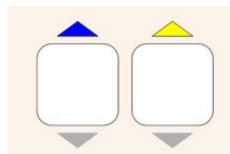
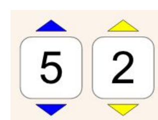
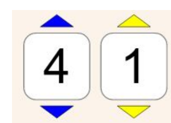
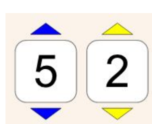
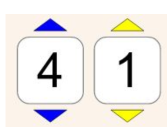
First Name:______________________ Last Name:______________________ Class:________

## Activity Sheet: Making Green Paint
## *with Proportion Playground*

**Learning Objectives - Students will be able to:**

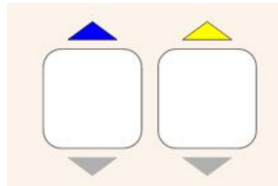
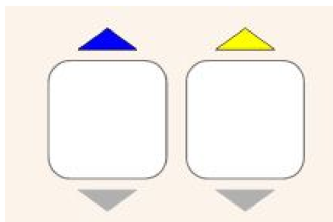
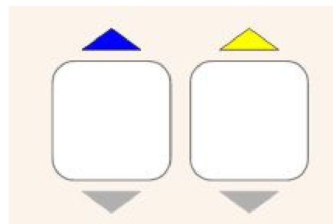
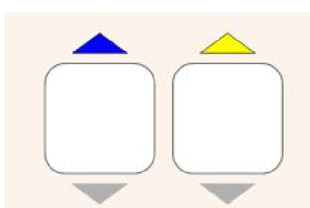
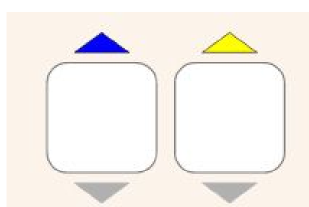
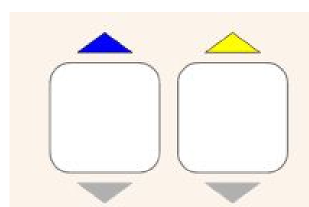
- Describe the difference between absolute (additive) and relative (multiplicative) thinking using informal language.

**Questions:**

| Question | Your answers |
|---|---|
| **#1** | **Write** the number of blues and the number of yellows that make the "greenest" paint.<br><br>Are there any other ways to make the "greenest" paint? Explain. |
| **#2** | Explain any patterns you notice in the numbers of blues and yellows.<br><br> |
| **#3** | **Predict** which statement you think will be true.<br>☐ They will be the *same* shade of green<br>☐ The  will be more green<br>☐ The  will be more green<br>**Explain** why you predict this: |

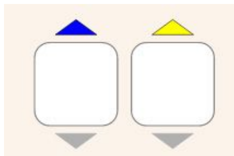
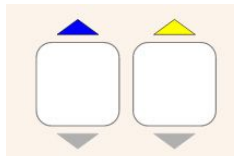
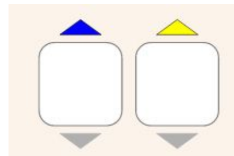
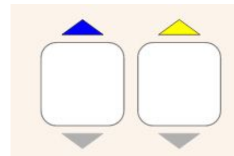
First Name:______________________ Last Name:______________________ Class:________

# Activity Sheet: Making Green Paint
## *with Proportion Playground*

| | |
|---|---|
| **#4** | **Write** the number of blue units and yellow units you used to make your favorite color.<br><br>**Pair #1:**  |
| **#5** | **Write** the number of blue units and yellow units on both sides.<br><br> <br><br>**Pair #1** **Pair #2**<br><br>**Complete the sentence:** *When I look at the two sets of numbers, I notice that…* |
| **#6** | **Write** Pair #1 and #2 again in the spaces below.<br>**Then** write the new pair of numbers in Pair #3.<br><br>**new!**<br><br>  <br><br>**Pair #1** **Pair #2** **Pair #3**<br><br>**Complete the sentence:** *When I look at the three sets of numbers, I notice that…* |

First Name:______________________ Last Name:______________________ Class:________

# **Activity Sheet: Making Green Paint**
## ***with Proportion Playground***

| | |
|---|---|
| **#7** | **Write** Pair #1,#2, and #3 again in the spaces below.<br>**Write** a new pair of numbers in Pair #4. <u>*Do not use the simulation to check before you write the new pair!*</u><br>**new!**<br> **Pair #1**  **Pair #2**  **Pair #3**  **Pair #4** |
| **#8** | Check if your 4th pair of numbers also makes you favorite color.<br><br>**Complete the sentence:** *The four pairs of numbers make the same color because…* |

**Reflection:**

| |
|---|
| *I am most proud of …* |

**Proportion Playground Paint Splat Activity Sheet**

---

Learning Goals

- Students will be able to create equivalent ratios.
- Students will be able to compare unequal ratios in a real-world context involving concentration levels.

---

## PART A: EXPLORE

1) Create your favorite shade of green.

Ratio of blue to yellow:

:

2) How many different ways can you create your favorite shade of green?

:     :     :     :

3) What do you notice about the ratios from #2?

STOP Discuss your answer for #3 with a partner before you move on.

---

## PART B: PREDICT

** Make sure you have switched to the PREDICT section of the sim and are using the black and white paint. **

4) BEFORE you use the sim, make a prediction. Then use the sim to fill out the actual column.

| | | |
|---|---|---|
| 3 5     8 10 | PREDICTION:<br>_____ left is darker<br>_____ right is darker<br>_____ both are the same shade. | ACTUAL:<br>_____ left is darker<br>_____ right is darker<br>_____ both are the same shade. |
| 3 2     6 4 | PREDICTION:<br>_____ left is darker<br>_____ right is darker<br>_____ both are the same shade. | ACTUAL:<br>_____ left is darker<br>_____ right is darker<br>_____ both are the same shade. |
| 3 6     4 9 | PREDICTION:<br>_____ left is darker<br>_____ right is darker<br>_____ both are the same shade | ACTUAL:<br>_____ left is darker<br>_____ right is darker<br>_____ both are the same shade |

STOP Discuss with your partner: What strategies can you use to predict which paint splat will be the darker shade?

5) Use your strategies from #4 to rank the paint mixtures from lightest to darkest. Try first WITHOUT using the sim. Later, you can use the sim to check your work.

| | | |
|---|---|---|
| **Mixture A** 2 3<br><br>**Mixture B** 2 8<br><br>**Mixture C** 3 6<br><br>**Mixture D** 4 4<br><br>**Mixture E**<br>You create it!<br><br><u>Challenge:</u> Create Mix E such that it is the middle in the list from lightest to darkest. | Lightest: __________<br><br>__________<br><br>__________<br><br>__________<br><br>Darkest: __________ | Explain or show work to justify your answer. |

STOP Pause for the whole-class discussion. Be prepared to explain the strategies you used in #5.

6) For mixtures A, B, C, and D in #5, write a fraction to describe black balloons to total balloons.

| | Mixture A | Mixture B | Mixture C | Mixture D |
|---|---|---|---|---|
| $\frac{\text{\# black balloons}}{\text{Total \# balloons}}$ | | | | |

7) Place the fractions from #6 on the number line below.


How does the number line help you confirm your answer to #5?