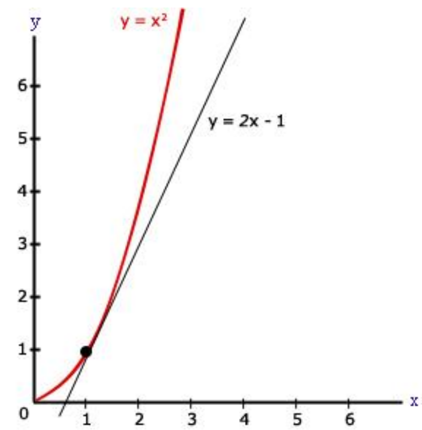
CHAPTER 6:

THE TANGENT PLANE

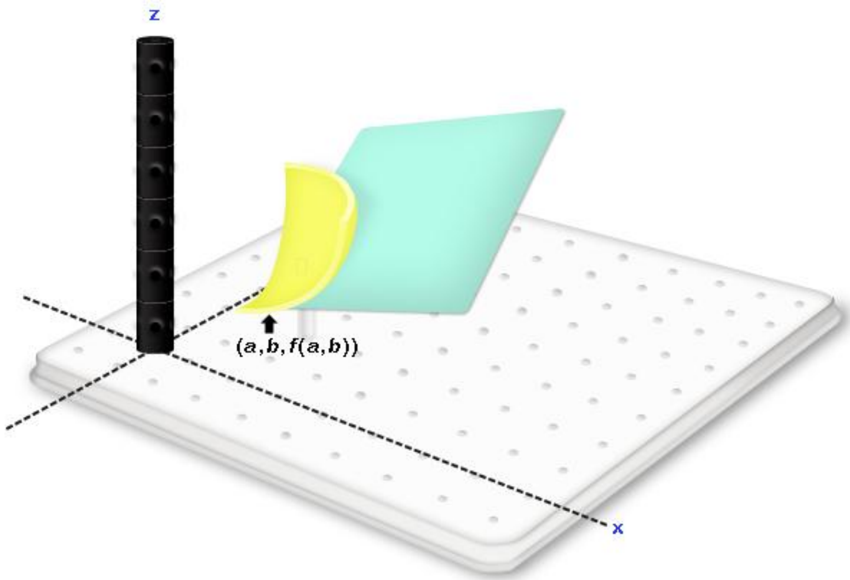
6.1 INTRODUCTION TO THE TANGENT PLANE

The following diagram contains the graph of the function $y = x^2$ and the graph of its tangent line at the point (1, 1)



With these graphs, it can be seen that there is a small region near the point (1, 1) where the graph of the tangent line appears identical to the graph of $y = x^2$. Correspondingly, the tangent line can frequently be used to approximate the behavior of the graph near the point to which the line is tangent.

In three dimensions, the tangent plane plays a similar role. We have seen that it is an easily obtainable tool. When discussing derivatives of functions of two variables. It is important to note that we can obtain the tangent plane to a surface very easily.

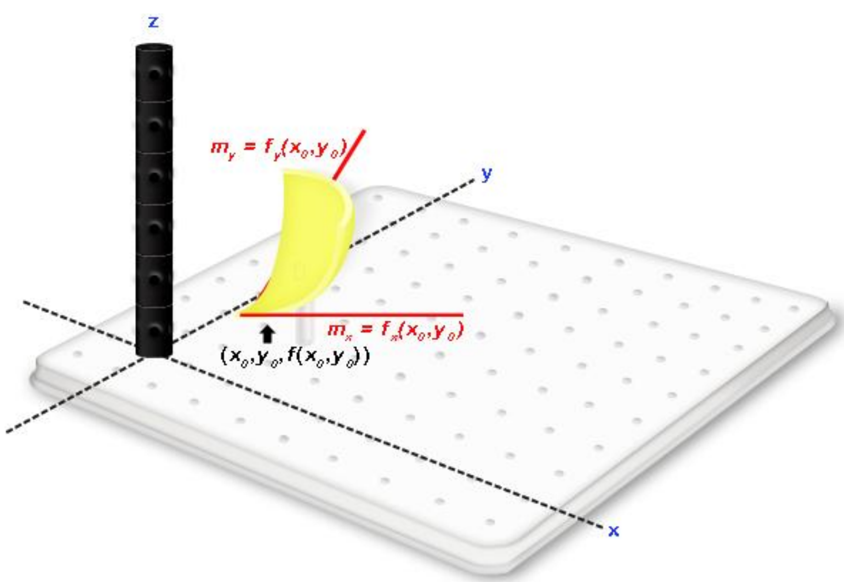


With this graph, we can see that, as with the tangent line in 2-D, there is a small region near the point (*a*, *b*, *f* (*a*, *b*)) where the behavior of the tangent plane and the behavior of the surface $z = f(x, y)$ appear identical.

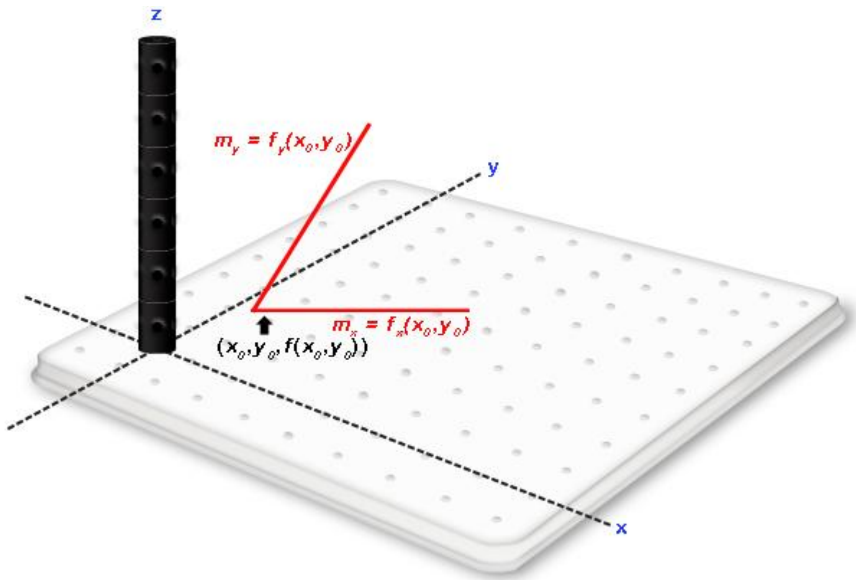
FINDING THE TANGENT PLANE

As was discussed in the section on planes, a point, the slope in the *x* direction and the slope in the *y* direction are sufficient information to identify and obtain the formula for a plane.

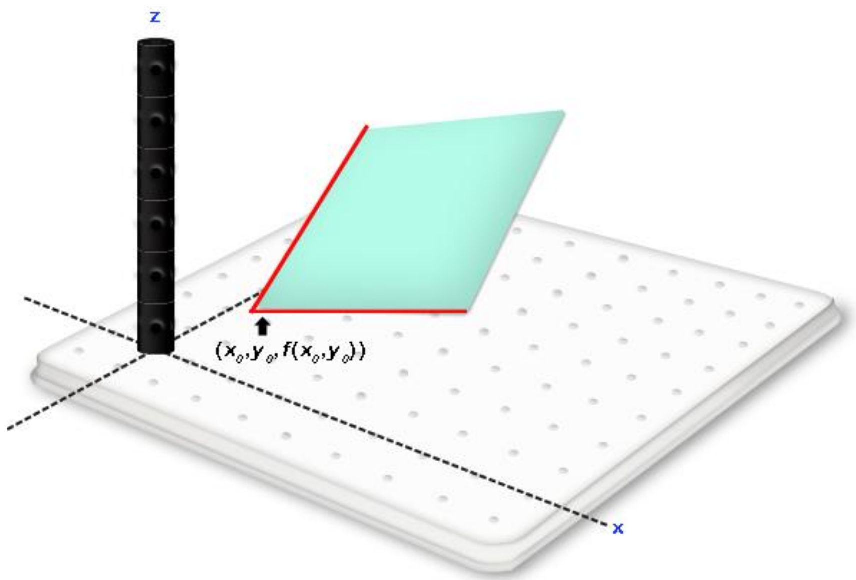
Given a surface $z = f(x, y)$ and a point (x_0, y_0), the tangent plane with $m_x = f_x(x_0, y_0)$ and $m_y = f_y(x_0, y_0)$. And the point $(x_0, y_0, f(x_0, y_0))$ will lie on the tangent plane.



As we now have a point on the tangent plane and two of its slopes, we no longer need the surface.



We can now use the two slopes and the point to obtain the tangent plane.



Hence, obtaining the formula and the geometric representation for the tangent plane is quite straightforward and correspondingly the tangent plane is an easily obtainable tool.

6.2 DIFFERENTIALS

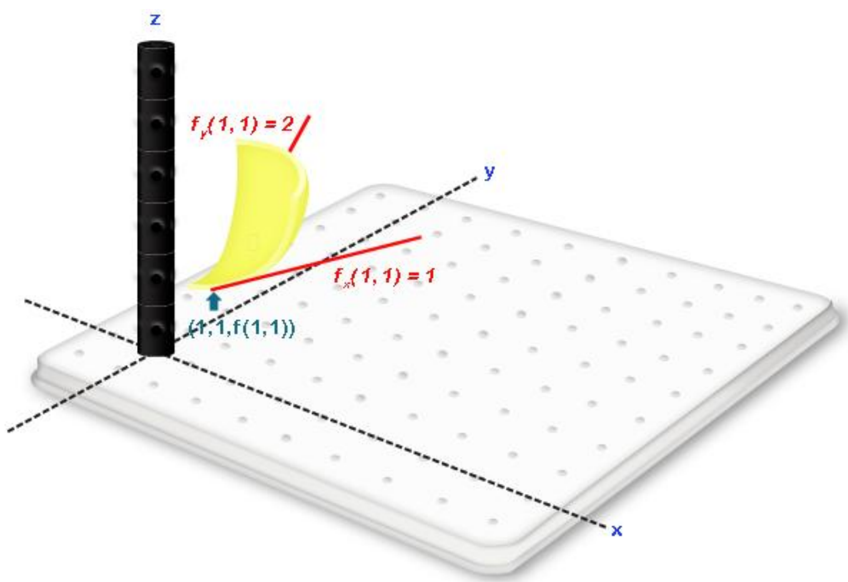
We have spoken of *rise* in a surface as Δz as we move from one point to another. We used to obtain approximations for partial derivatives. However, when we are using the tangent plane as an approximation, we can consider *rise* as we move along the tangent plane as well. To differentiate between whether we are moving on the surface $z = f(x, y)$ or the tangent plane to the surface at some given point, the following convention will be used:

- When moving along a surface, $z = f(x, y)$, the change in height will be referred to as Δz.
- When moving along the tangent plane to a surface $z = f(x, y)$ at some given point, the change in height will be referred to as the differential *dz*.

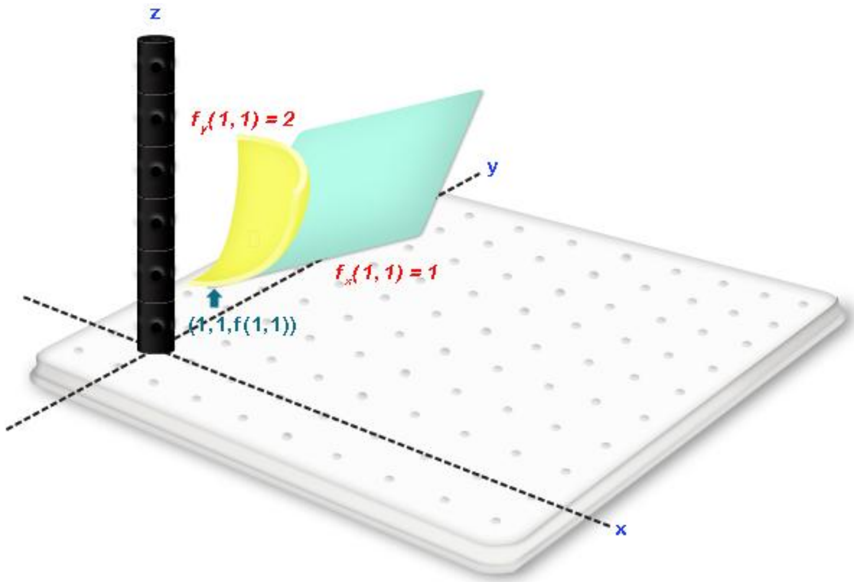
Example Exercise 6.2.1: Given the surface $z = .5x^2 + y^2$, find *dz* if $\Delta x = 2$ and $\Delta y = 3$ and the tangent plane at the point (1, 1, 1.5) is it to be used to approximate the surface.

Solution:

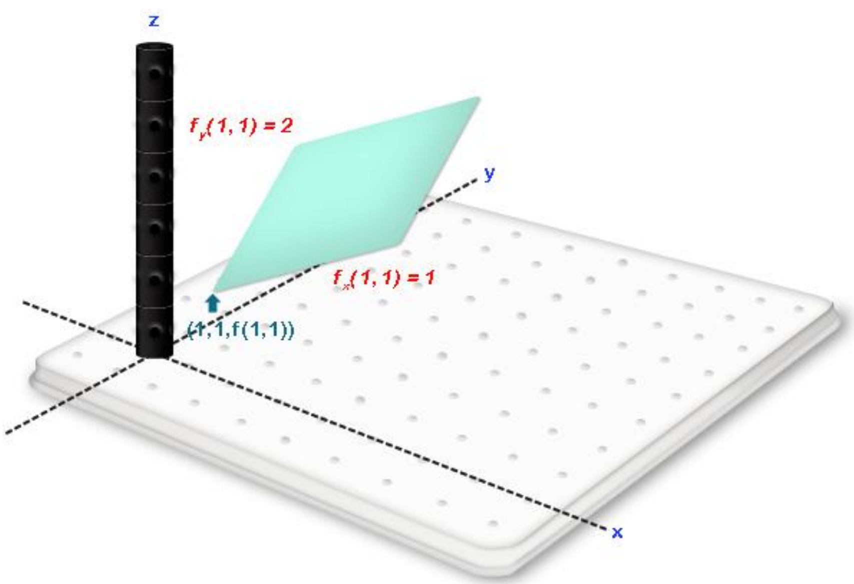
1. Obtain the tangent plane to the surface:



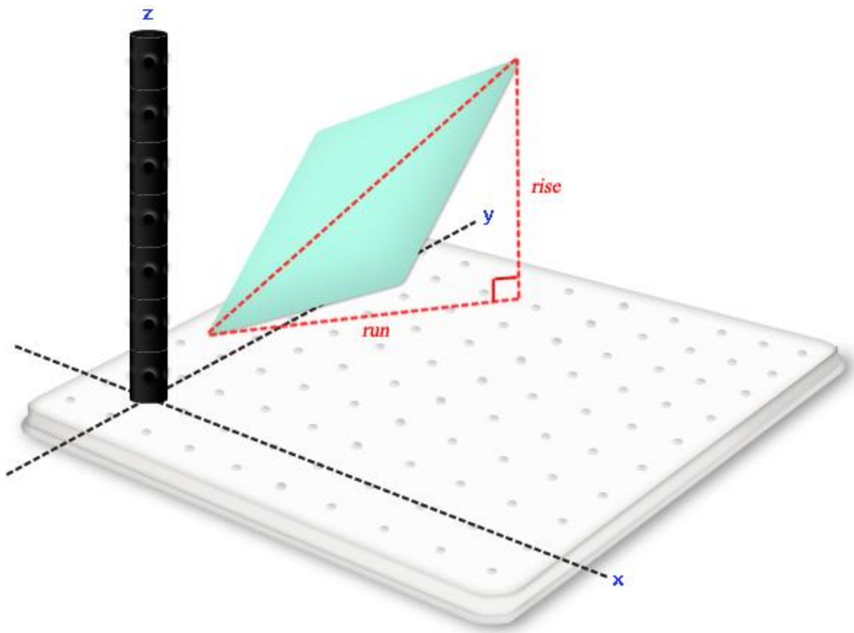
Using the point and the two slopes, the tangent plane will be:



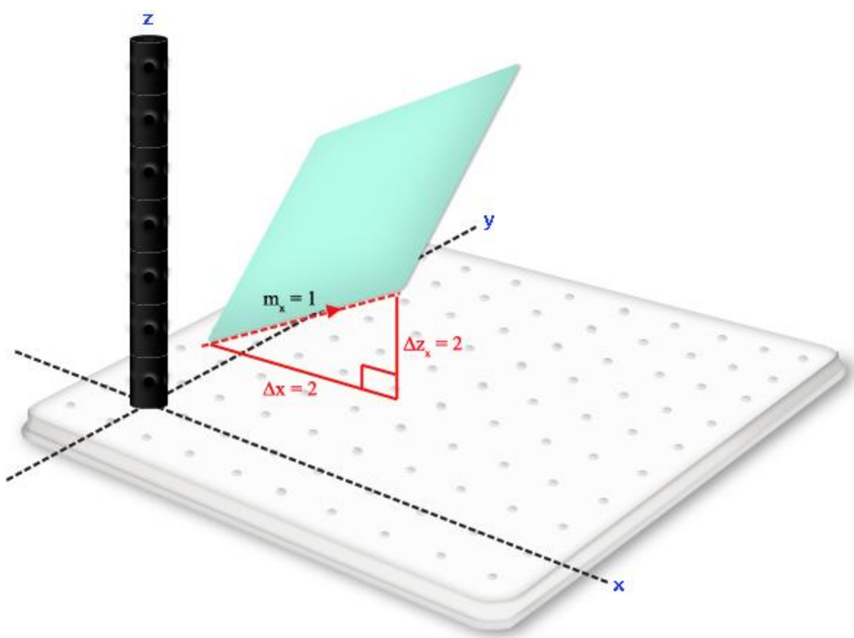
As we are seeking the differential, the surface can be discarded and we can pay attention solely to the tangent plane in order to obtain the desired change in height. The two data we use on the tangent plane are $m_x = f_x(1,1) = 1$ and $m_y = f_y(1,1) = 2$.



2. Identify the right triangle whose *rise* and *run* are associated with these slopes.

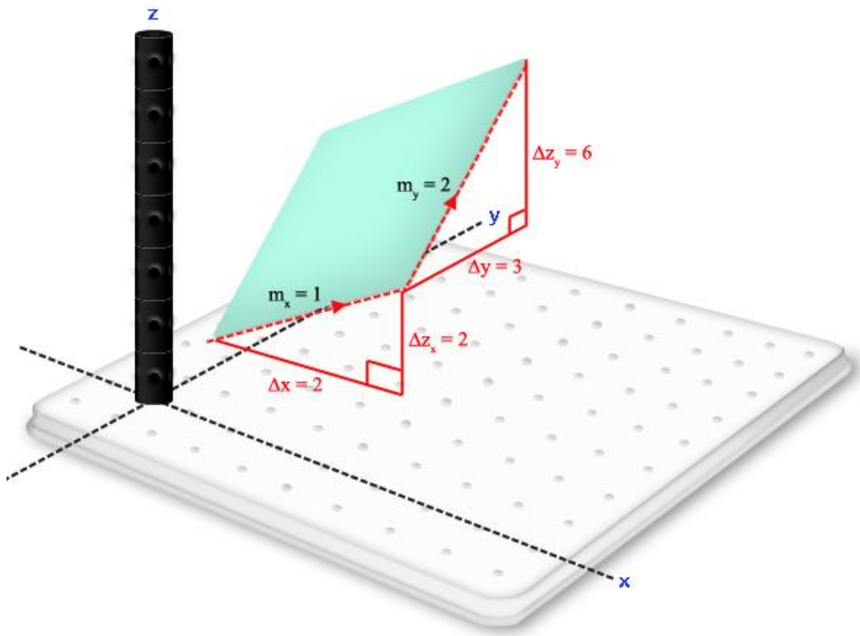


3. Find the change in height associated with the *x* direction:



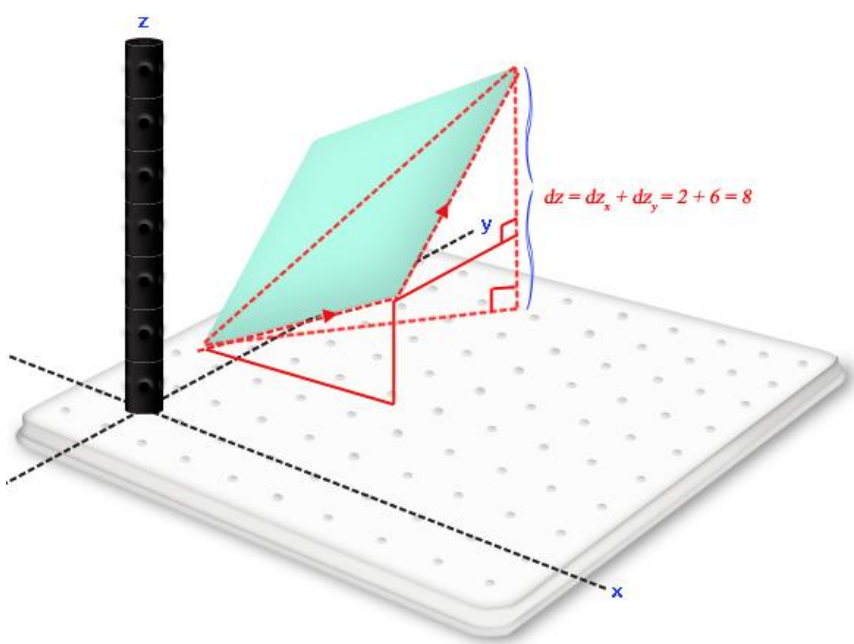
$$\Delta x \ = 2, m_{\mathrm{x}} = 1, x = 1 \rightarrow dz_{\mathrm{x}} = 2$$

4. Find the change in height associated with the *y* direction:



$$\Delta y = 3, m_{\mathrm{y}} = 2 \rightarrow dz_{\mathrm{y}} = 6$$

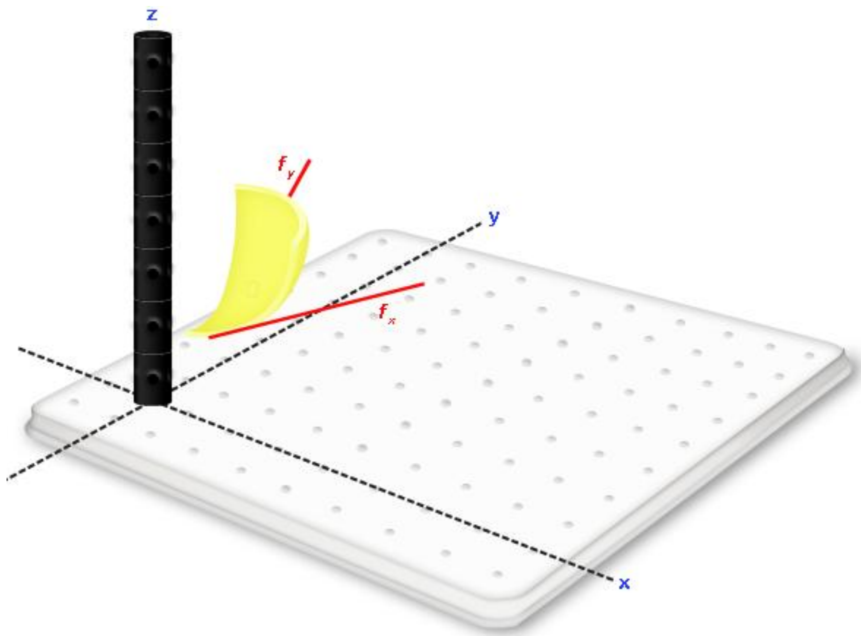
5. Find the overall change in height: Taking the change in height associated with movement of 2 units in the x direction and 3 units in the y direction and adding them to yield the total change in height associated with this movement yields $dz = dz_{\mathrm{x}} + dz_{\mathrm{y}} = 8$.



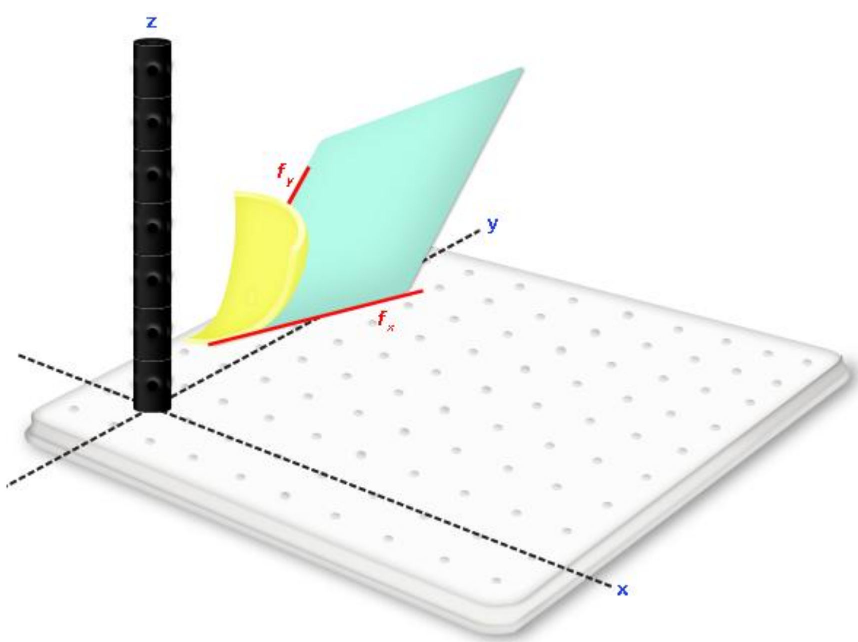
Generalization

Given a function $z = f(x, y)$ and a point $(x,y)=(a,b)$, in order to find the differential dz associated with movement in the x direction dx and movement in the y direction dy, we can follow the following steps.

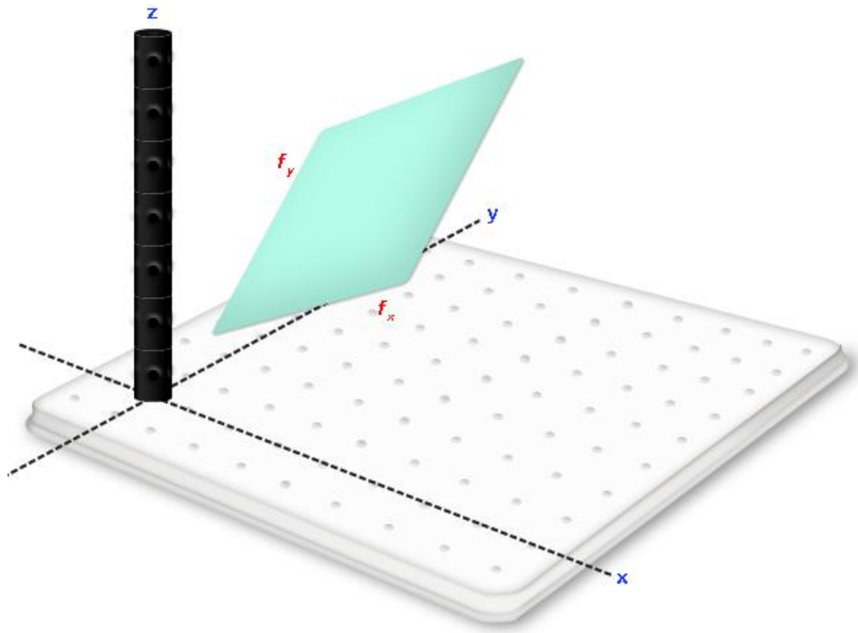
1. Obtain the tangent plane to the surface



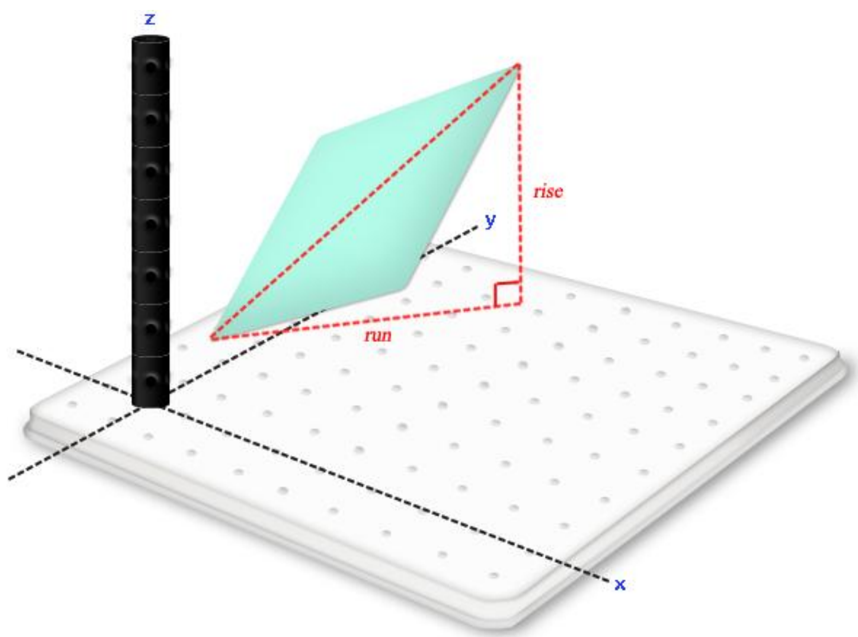
Using the point and the two slopes, the tangent plane will be



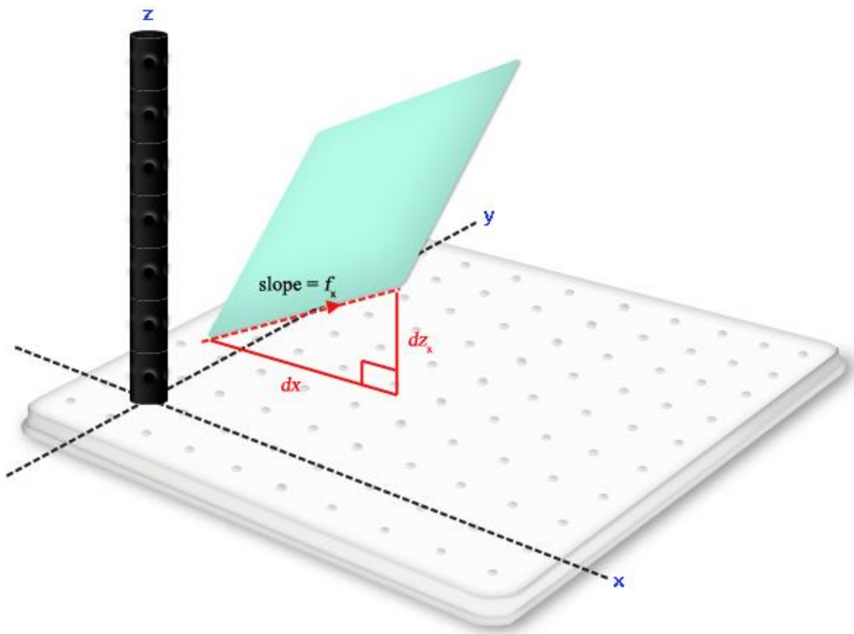
As, we are seeking the differential, the surface can be discarded and we can pay attention solely to the tangent plane in order to obtain the desired change in height.



2. Identify the right triangle whose *rise* and *run* are associated with these slopes.

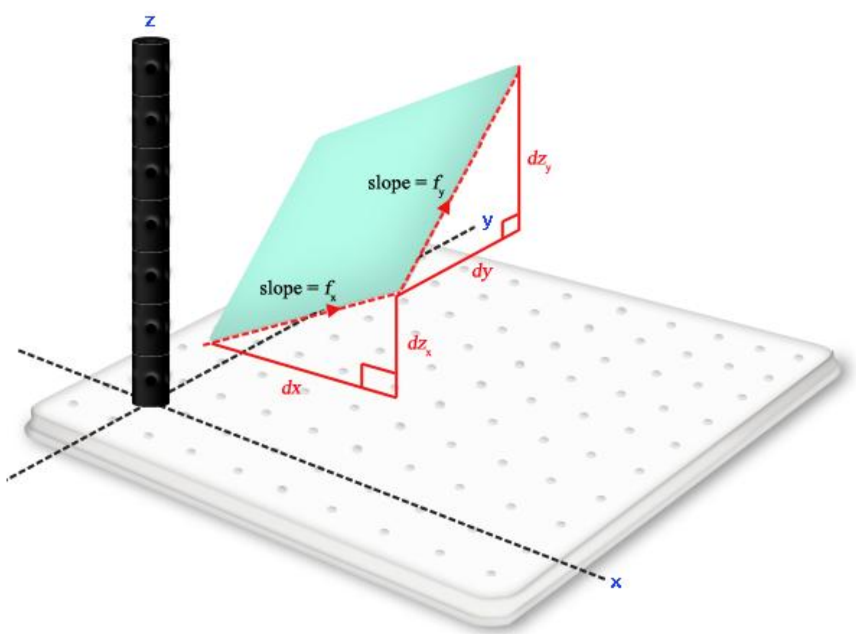


3. Find the change in height associated with the *x* direction:



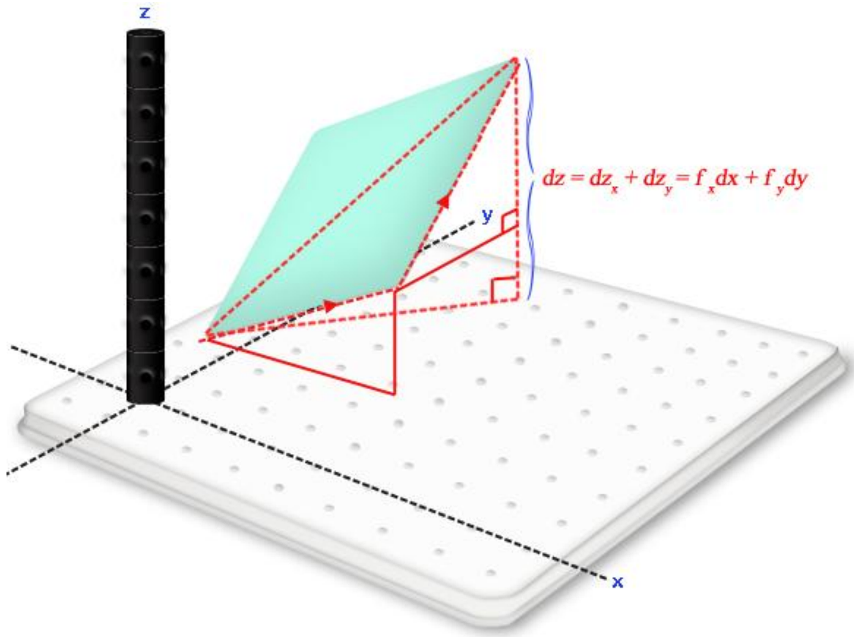
The *rise* in the *x* direction $dz_x = f_x(a,b)\, dx$

4. Find the change in height associated with the *y* direction:



The rise in the *y* direction $dz_y = f_y(a,b)dy$

5. Taking both documents of the change in height on the tangent plane or the differential is $dz = dz_x + dz_y = f_x\, dx + f_y\, dy$



6.3 DIRECTIONAL DERIVATES

DEFINITION

$D_{<u1,u2>} f(x_0, y_0)$ is defined as the slope of the tangent line to $z = f(x, y)$ at the point (x_0, y_0) and in the direction $<u_1,u_2>$.

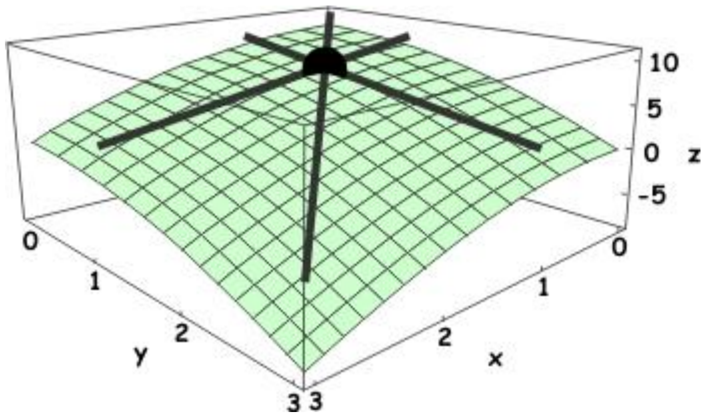
Example

$D_{<1, 0>} f(x_0, y_0) = D_{<2, 0>} f(x_0, y_0) = f_x(x_0, y_0)$ and $D_{<0, 1>} f(x_0, y_0) = D_{<0, 2>} f(x_0, y_0) = f_y(x_0, y_0)$.

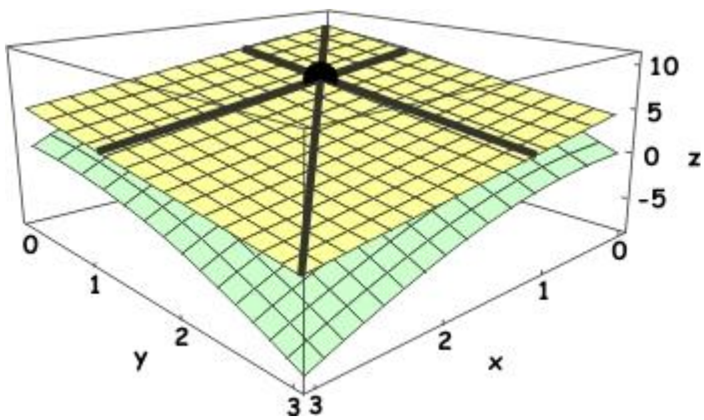
These are directional derivatives we can obtain with the current knowledge we have available. If we wish the slopes in other directions, we need to further develop the knowledge we have.

USING THE TANGENT PLANE TO FIND DIRECTIONAL DERIVATES

By definition, $D_{<u1,u2>} f(x_0, y_0)$ is defined as the slope of the tangent line to $z = f(x, y)$ at the point (x_0, y_0) and in the direction $<u_1,u_2>$. In the following diagram we see various tangent lines to a surface at a given point.



In the following diagram we can see that all of the tangent lines, irrespective of the direction lie on the tangent plane to the surface at the point (x_0, y_0).



Hence, we can conclude that the directional derivative of a surface $z = f(x, y)$ at a point (x_0, y_0) is equal to the directional slope of the tangent plane at that point.

Hence with this and data from previous sections, we can conclude the following:

- The tangent place to a surface is easily obtained.
- If we know the slopes m_x and m_y of a plane, we can obtain the slope of a plane in any direction.
- The directional derivative of a surface is the same as the directional slope of the tangent plane.

Hence, we have the following general procedure with which we can obtain directional derivatives.

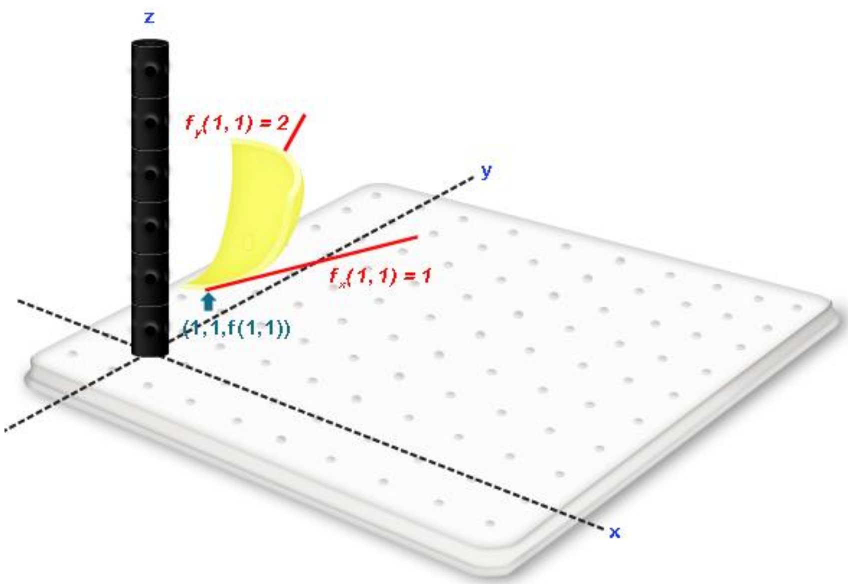
1. Obtain the tangent plane to the surface $z = f(x, y)$ at the point where the directional derivative s desired.

2. Obtain the directional slope of the tangent plane in the indicated direction and it will be the same as the directional derivative of the surface.

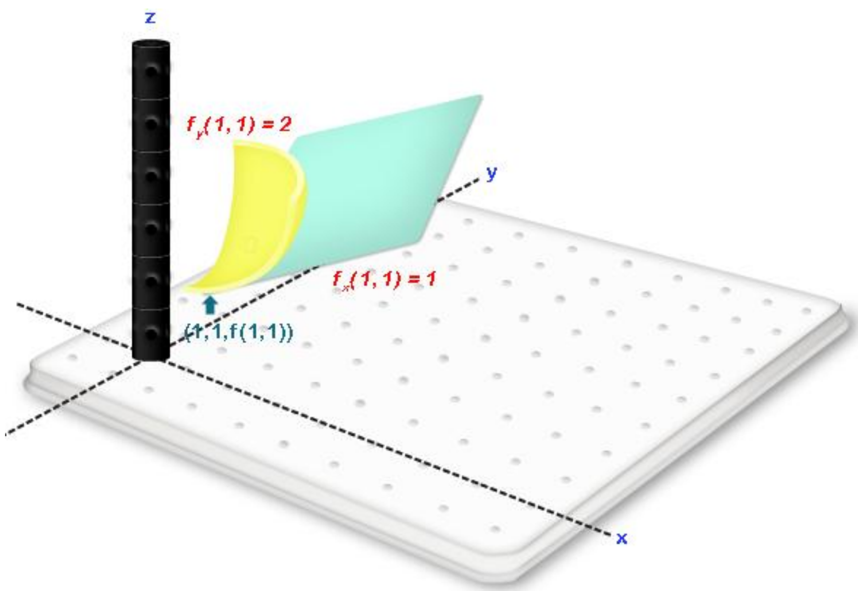
Example Exercise 6.3.1: Given the surface $z = .5x^2 + y^2$, find $D_{<2,\,3>}\,f\,(1, 1)$.

Solution:

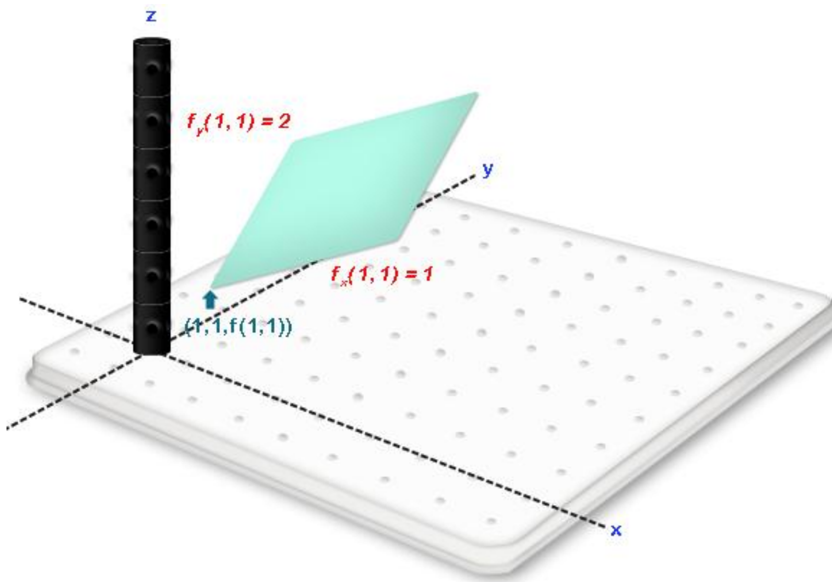
1. Obtain the tangent plane to the surface



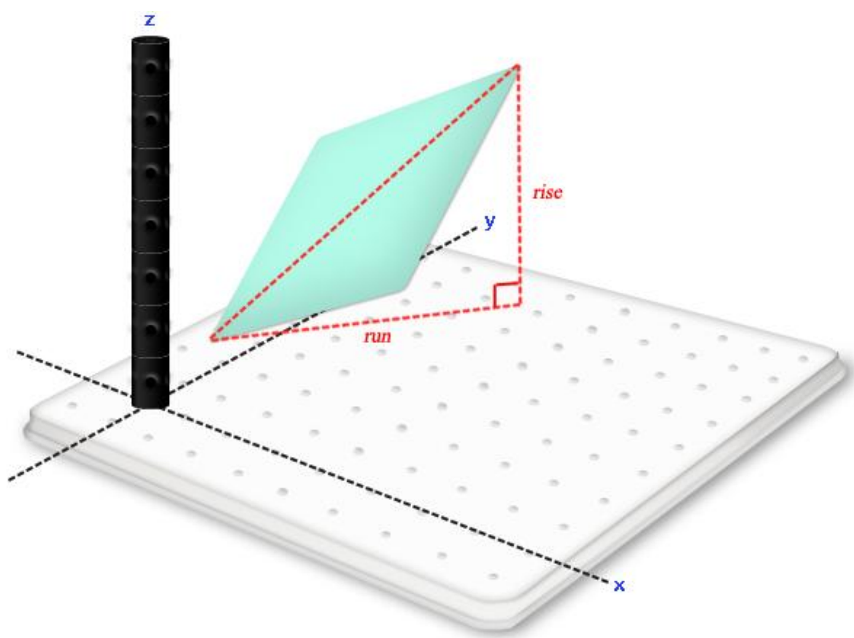
Using the point and the two slopes, the tangent plane will be



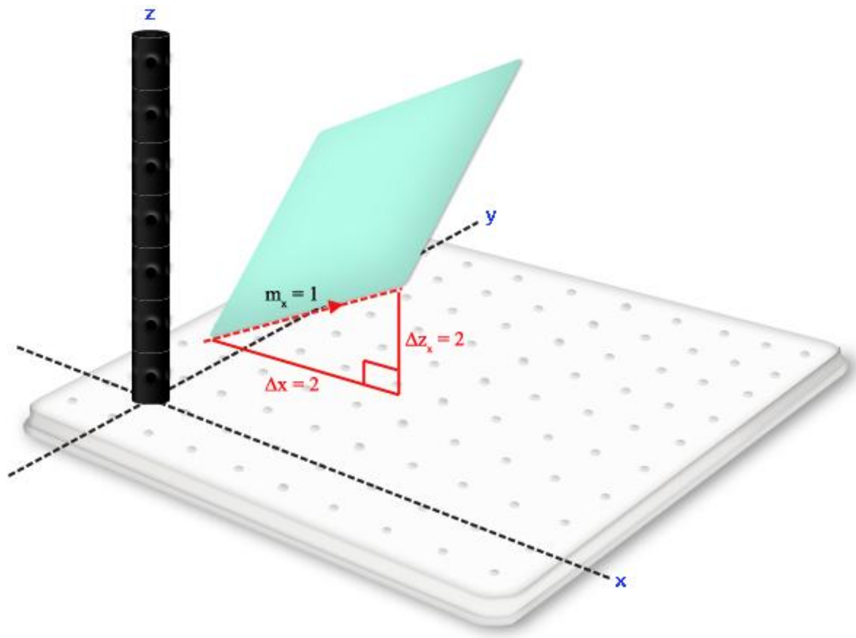
The surface can be discarded as the directional derivative of the function will be the direction slope of the tangent plane we have just obtained.



2. Identify the right triangle whose *rise* and *run* are associated with these slopes.

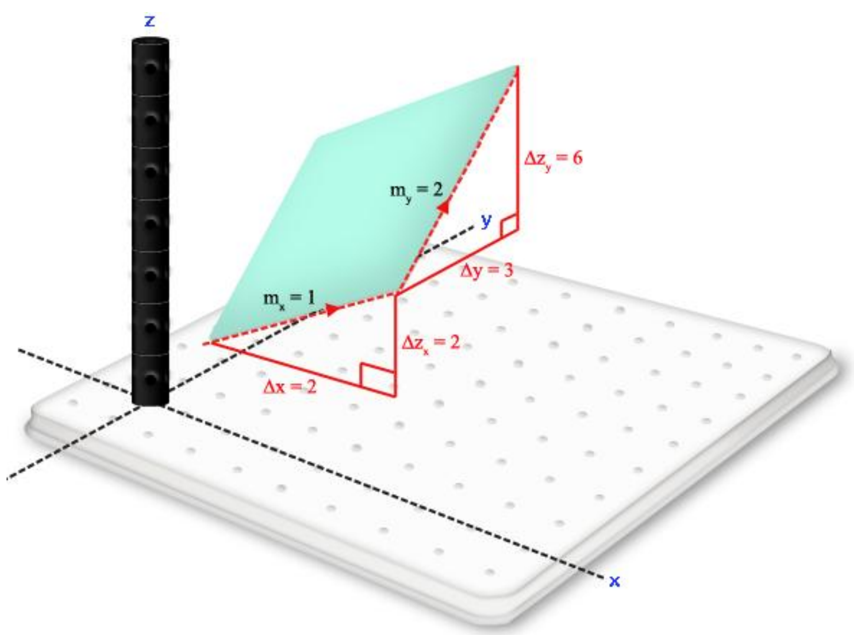


3. Find the change in height associated with the *x* direction of <2, 3>:



$$D_x = 2, m_{\mathrm{x}} = 1 \rightarrow dz_{\mathrm{x}} = 2$$

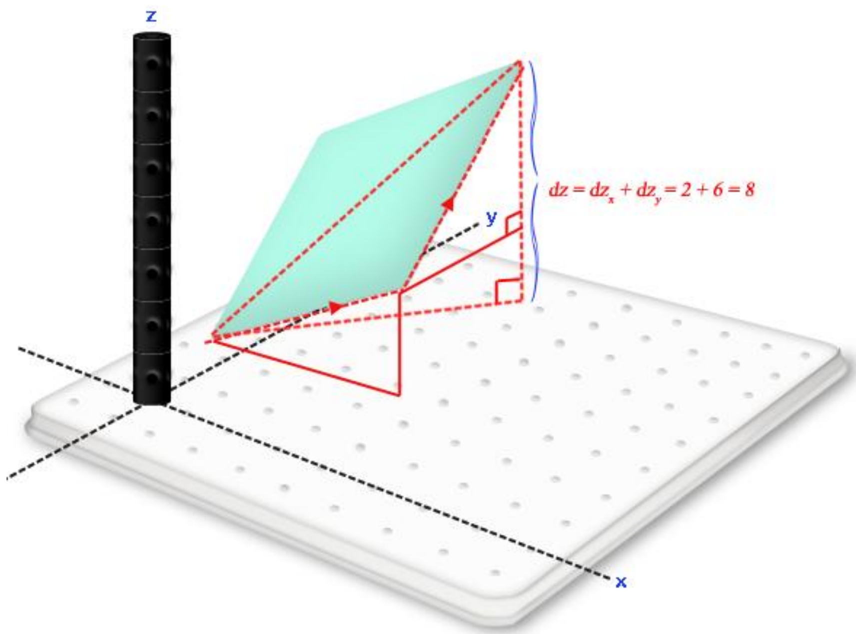
4. Find the change in height associated with the *y* direction <2, 3>:



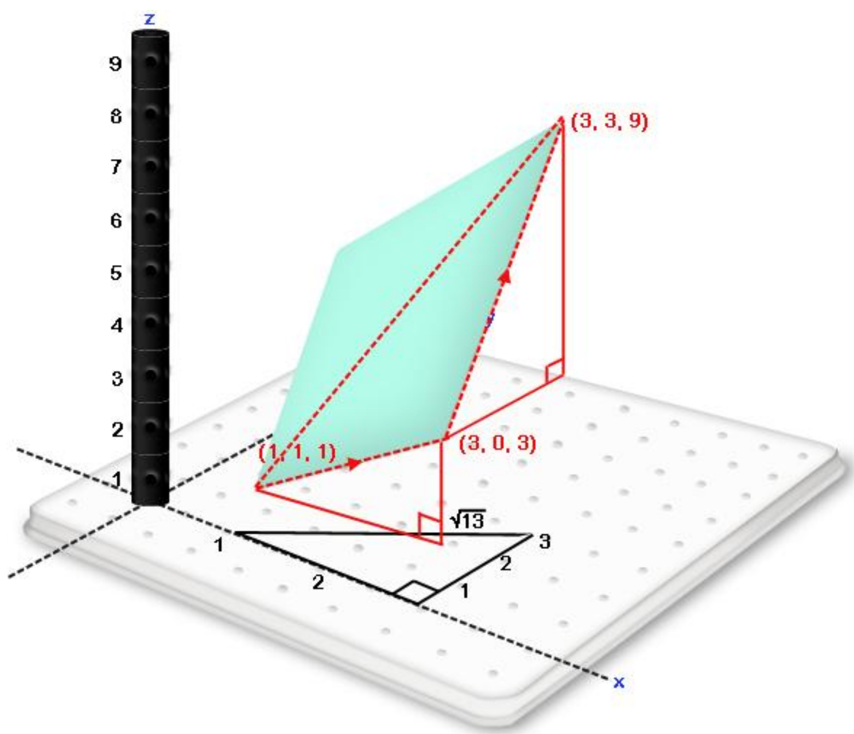
$$D_y = 3, m_{\mathrm{y}} = 2 \rightarrow dz_{\mathrm{y}} = 6$$

5. Find the overall change in height:

Taking both components of the change in height, the total difference in height is $dz = dz_x + dz_y = 8$.



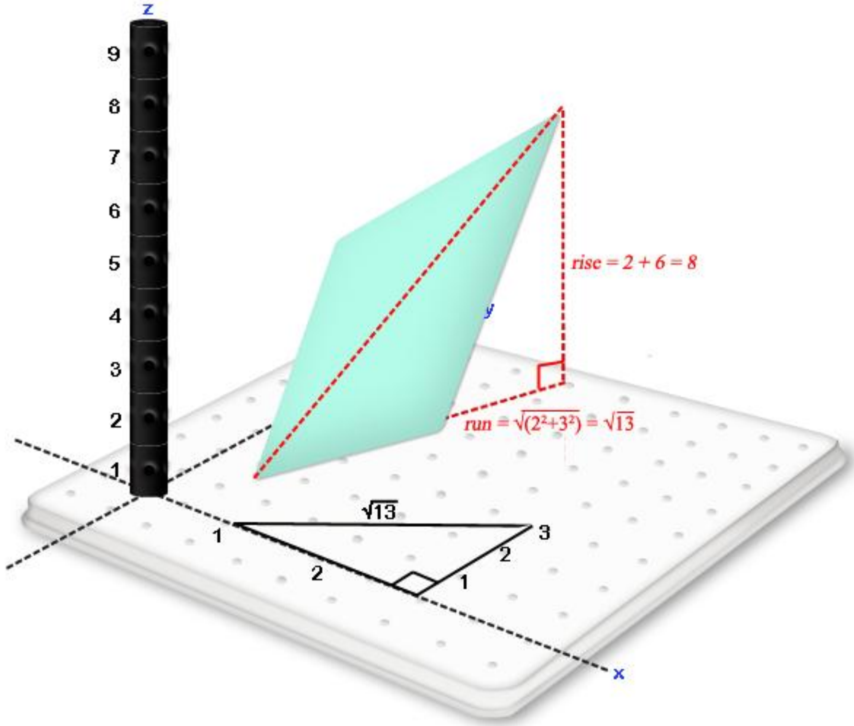
6. Obtain the *run* for the movement associated with <2, 3> i.e., $D_x = 2$, $D_y = 3$



Using Pythagoras and the above right triangle, $run = \sqrt{2^2 + 3^2} = \sqrt{13}$

7. Obtain the slope:

$$\frac{rise}{run} = \frac{8}{\sqrt{13}}$$



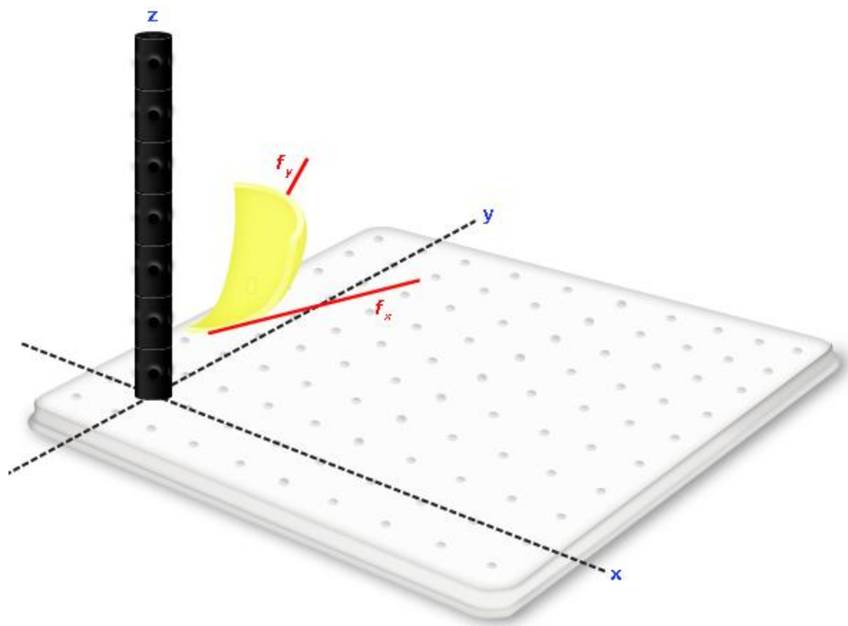
Conclusion: The directional slope of the tangent plane is the same as:

$$D_{\langle 2,3\rangle} f(1,1) = \frac{8}{\sqrt{13}}$$

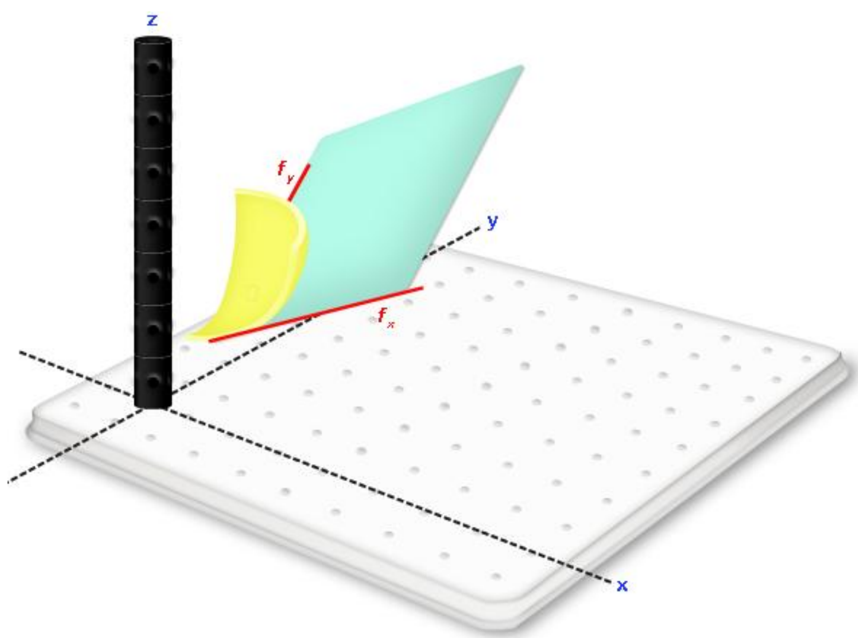
<u>Generalization</u>

Given a function $x = f(x, y)$ and a point (x_0, y_0), we can find $D_{<u1,u2>} f(x_0, y_0)$ through the following steps:

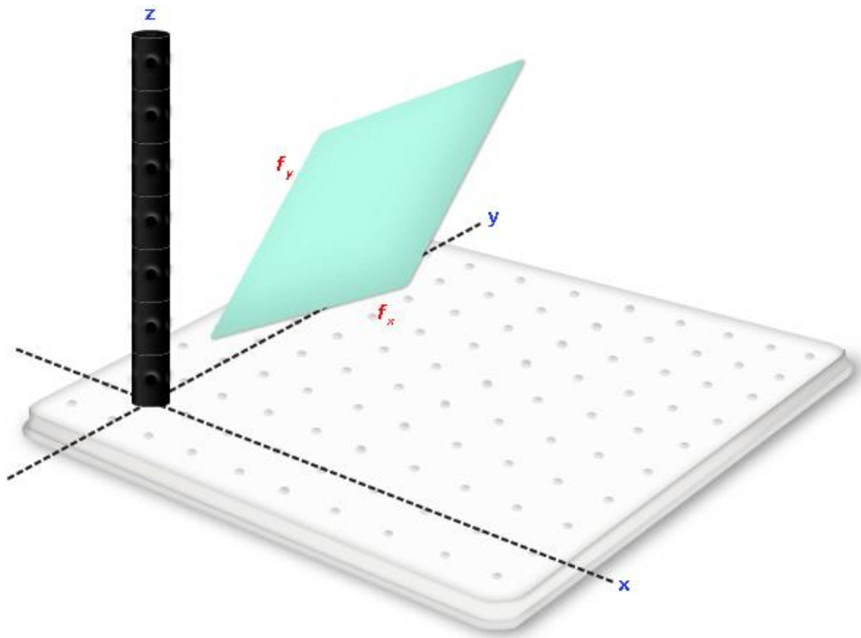
1. Obtain the tangent plane to the surface:



Using the point and the two slopes, the tangent plane will be

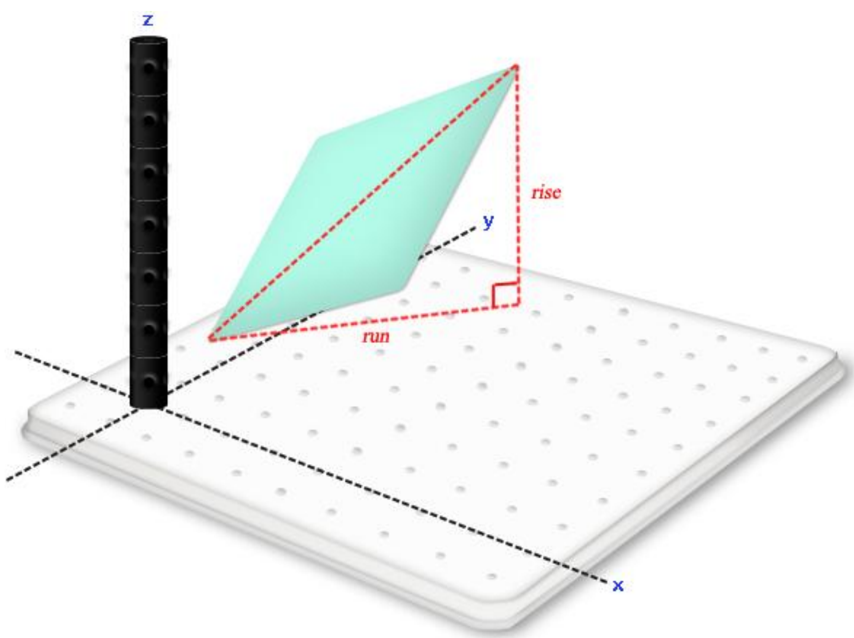


As we are seeking the differential, the surface can be discarded and we can pay attention solely to the tangent plane in order to obtain the desired change in height.



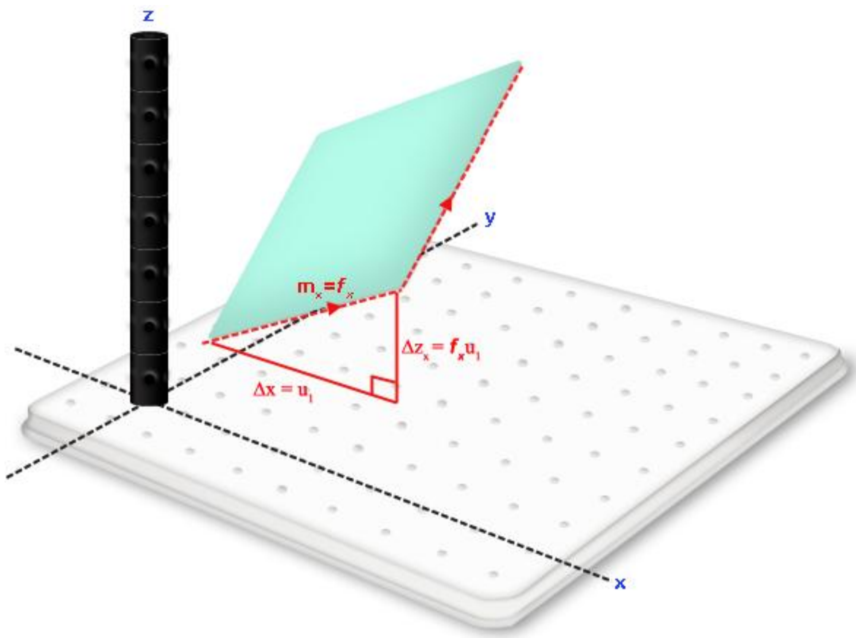
Notice that we have obtained the tangent plane, $D_{<u1,u2>} f(x_0, y_0)$ is equal to the slope on the tangent plane in the direction $<u_1,u_2>$.

2. Identify the right triangle whose *rise* and *run* are associated with these slopes.



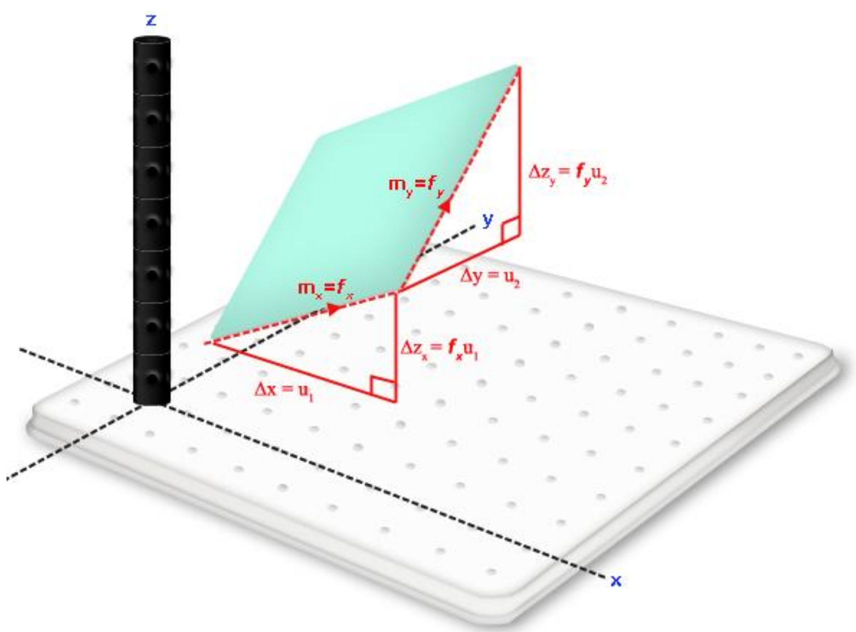
3. Find the *rise* in the tangent plane

   Rise in the *x* direction



$$\Delta x = \mathrm{u}_1, \mathrm{m}_x = f_x \to dz_x = f_x \mathrm{u}_1$$

Rise in the *y* direction



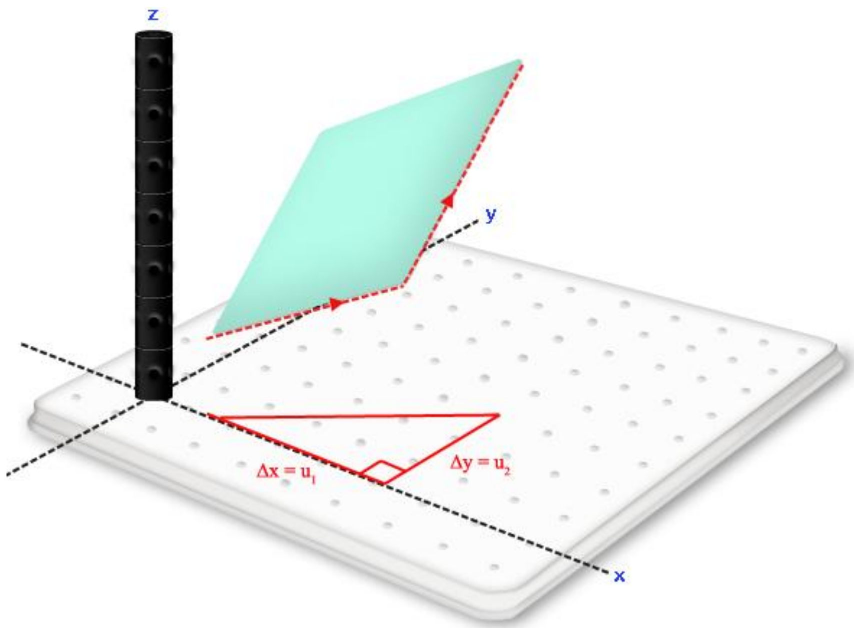
$$\Delta y = \mathrm{u}_2, \mathrm{m}_y = f_y \to dz_y = f_y \mathrm{u}_2$$

Total *rise*:

$$dz = dz_x + dz_y = f_x \mathrm{u}_1 + f_y \mathrm{u}_2$$

4. Obtain the *run* for

$$\Delta x = \mathrm{u}_1, \Delta y = \mathrm{u}_2$$



$$run = \sqrt{\mathrm{u}_1{}^2 + \mathrm{u}_2{}^2}$$

5. The directional slope of the tangent plane is the same as the directional derivative

$$D_{\langle \mathrm{u}_1, \mathrm{u}_2 \rangle} f(x_0, y_0) = \frac{rise}{run} = \frac{f_x \mathrm{u}_1 + f_y \mathrm{u}_2}{\sqrt{\mathrm{u}_1{}^2 + \mathrm{u}_2{}^2}}$$

It is worth nothing that if $<u_1,u_2>$ is a unit vector then

$$D_{\langle \mathrm{u}_1, \mathrm{u}_2 \rangle} f(x_0, y_0) = \frac{rise}{run} = \frac{f_x \mathrm{u}_1 + f_y \mathrm{u}_2}{1} = f_x \mathrm{u}_1 + f_y \mathrm{u}_2$$